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Subgame-Perfect Equilibria in Stochastic Timing Games[☆]

Frank Riedel[a,b], Jan-Henrik Steg[a,*]

[a] Center for Mathematical Economics, Bielefeld University, Germany
[b] Faculty of Economic and Financial Sciences, University of Johannesburg, South Africa

Abstract

We develop a notion of subgames and the related notion of subgame-perfect equilibrium – possibly in mixed strategies – for stochastic timing games. To capture all situations that can arise in continuous-time models, it is necessary to consider stopping times as the starting dates of subgames. We generalize Fudenberg and Tirole's (1985) mixed-strategy extensions to make them applicable to stochastic timing games and thereby provide a sound basis for subgame-perfect equilibria of preemption games. Sufficient conditions for equilibrium existence are presented, and examples illustrate their application as well as the fact that intuitive arguments can break down in the presence of stochastic processes with jumps.

Keywords: timing games, stochastic games, mixed strategies, subgame-perfect equilibrium, continuous time, optimal stopping
JEL: C61, C73, D21, L12
2000 MSC: 60G40, 91A25, 91A40, 91A55

1. Introduction

Timing plays a crucial role in economics, be it in terms of the right time to enter a market, to stop an experiment, or to initiate a new monetary policy, to name just a few important issues. Such timing issues are often best formulated in continuous time because then powerful analytical tools allowing for elegant

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[*] Corresponding author

solutions and providing additional insight into the structure of timing decisions become available. Continuous time may also be more appropriate for problems in which the strict order of moves that discrete time dictates appears as an arbitrary imposition. However, modeling dynamic information and interaction in continuous time is conceptually much more challenging than in discrete time. Familiar principles from discrete time should serve as analogies, but they cannot be transferred directly.

In this paper we establish a framework to analyze timing games dynamically in continuous time, possibly under uncertainty about some exogenous state of the world. Therefore we address issues concerning two kinds of dynamic information in continuous time: information about moves of strategically acting agents and information about the exogenous state of the world.

First, from a game-theoretic perspective, it is well known that modeling interaction in continuous time leads to fundamental problems. If one tries to mimic discrete time by defining histories of actions at each time $t \in \mathbb{R}_+$ – and strategies depending on these histories – determining consistent outcomes quickly becomes impossible. The problem arises if a "move" by some player does not identify a specific "next" point in time at which another player is allowed to move. As continuous time is not well ordered, it would be necessary to impose such a structure to obtain an analog of the extensive form of discrete-time games.[1] These problems even pertain to timing games in which each player can only move once such that histories seem to be simple objects (see Simon and Stinchcombe, 1989, for many illuminating examples). Nevertheless, the fact that each player has only one move allows us to model some instantaneous reactions, but to avoid mutual conditioning of simultaneous moves. Therefore we adopt the following concept implicit in, e.g., Fudenberg and Tirole (1985) and Laraki et al. (2005).

The dynamic perspective on the game is composed of stacking *normal form* views taken in different situations in a consistent way. Situations are therefore distinguished by time, a *mode* which describes who has already moved, and the information about the exogenous state in our case. Normal form strategies are plans when to move that are conditional on the present mode. These strategies are, thus, only executed *if no one moves before.* Changes to the mode are observable instantaneously, and then the strategies of the remaining players for the new mode apply. This concept facilitates, e.g., that player 2 moves whenever player 1 moves, but not that player 2 moves only if player 1 *does not* move.[2]

A new normal form view is taken whenever a move could be scheduled. It is

[1] See, e.g., Alós-Ferrer and Ritzberger (2008) for problems in defining dynamic strategies that can be mapped to consistent outcomes. The authors propose an infinite generalization of classic trees and show that it is generally necessary for nodes to have clearly defined successors. Simon and Stinchcombe (1989) restrict continuous-time games to increasingly fine grids and bound the number of moves in order to obtain well-defined limit outcomes.

[2] In order to move only as an instantaneous reaction, player 2's strategy for the initial mode would be to never move; for the mode "player 1 has moved," player 2's strategy would be to move instantaneously.

then possible to consider arbitrary deviations – respectively revisions of plans – and to apply the dynamic programming principle during each mode.

Situations in which moves can (potentially) occur define the starts of subgames in discrete time. In continuous time we need to carefully consider the second kind of dynamic information – that concerning the exogenous state of the world – in order to define subgames. As before it is not sufficient to treat times $t \in \mathbb{R}_+$ individually, in contrast to other attempts in the literature. It is generally impossible to obtain well-specified plans of behavior for all contingencies in continuous time by only singling out information sets at different values of t – on which a move can occur or not – as is done in discrete time. Instead, as we argue, a definition of subgames needs to be based on the notion of *stopping times* in order to gain a complete and consistent model.

Normal form views of different subgames overlap substantially. We require strategies for different subgames to be consistent, such that the overall view can be understood analogous to the extensive form analysis in discrete time. When applied to discrete time, our approach becomes indeed equivalent.[3]

On the one hand, our new concept of subgames and subgame-perfect equilibria provides a complete and consistent mathematical foundation for timing games in continuous time (under uncertainty). On the other hand, our concept also reflects the economic structure of the players' decision problems more appropriately than earlier attempts by placing a strong emphasis on dynamic programming principles. This general framework also benefits the analysis of models with Markovian structure, like most applications of stochastic timing games, for which it is natural to use reduced-form value functions on the state space.[4] Thinking rigorously in terms of stopping problems helps in understanding equilibrium mechanics better and in providing complete arguments for equilibrium verification.

Indeed, we apply our framework to give sound footing to important basic results from the growing literature on strategic real options (e.g., from some seminal papers listed below). Therefore we also extensively discuss preemption issues, which may cause equilibrium nonexistence, even for mixed strategies. In addition to allowing players to randomize over plans of when to move, we define strategy extensions that facilitate endogenous coordination in order to support equilibria with mutual preemption.

More specifically, we generalize Fudenberg and Tirole's (1985) approach such that it can be applied to stochastic settings.[5] The approach is less ad hoc

[3] A consistent set of normal form strategies in discrete time can be fully characterized by the prescriptions for the respective first period, specifically by the probabilities with which the planned move times are or exceed the respective first period. Furthermore, any state-dependent (stopping) time in discrete time can be fully characterized by specifying for each period the event on which the time occurs.

[4] Dutta and Rustichini (1993) have a Markovian model and only consider equilibria in Markovian strategies. They first define a more general notion of subgames and strategies, but again based on deterministic dates.

[5] Thijssen et al. (2012) – and subsequently Boyarchenko and Levendorskiĭ (2014) – take a different route to adapting Fudenberg and Tirole's (1985) approach. First, they use *uncon-*

than are simple tie-breaking rules[6] and does not replace any outcomes with others, thereby leaving some risk of simultaneous moves, which is a key aspect of preemption. This approach furthermore aims at consistency with limits from discrete-time games with vanishing period lengths.[7] To accommodate standard asymmetric or stochastic models, we have to remove some of Fudenberg and Tirole's (1985) regularity properties and carefully revisit the determination of outcomes. Although defining the outcome for *any* profile of extended strategies takes up some space, it follows a clear intuition and leads to general standard patterns to support preemption equilibria. Our results can be used as a basis for the dynamic analysis of much richer models than before and can thereby extend both the applied and the theoretic literature.[8]

In order to embed preemption into more general subgame-perfect equilibria, we finally consider the complementary problem: verifying the optimality of waiting. Such problems are neglected in many real-option games. An intuitive sufficient condition for the optimality of waiting is that there is second-mover advantage and that the first-mover's payoff increases in expectation. We reveal that this argument is sensitive to jumps in the underlying processes. Specifically, we prove a general positive result for continuous processes. Afterwards we present an economic example involving random jumps for which the argument fails. The jumps are generated in the most standard way by a Poisson component.

1.1. Related literature

Strategic timing problems appear in an abundance of contexts, particularly in economics. There is a vast amount of literature on this classic topic, and we thus only name a few works that are most related to ours for certain reasons. On the one hand there is the literature on mainly deterministic timing games in continuous time that is inspired by a wide range of applications, such as preemption problems in economics (e.g., Fudenberg and Tirole, 1985; Hendricks and Wilson, 1992) or wars of attrition in biology or economics (e.g., Hendricks et al., 1988). These models are often very stylized with systematic first or

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ditional strategies, which do not depend on whether the respective other player has already moved. Second, they force each player's strategy and distribution of move time to be identical by imposing a joint restriction on feasible *profiles* of strategies. Our strategies are conditional on the move history, and we allow each player to choose any strategy from the individually feasible set and then determine induced outcome distributions.

[6] A commonly used rule is coin tossing. See, e.g., Grenadier (1996) or Hoppe and Lehmann-Grube (2005).

[7] Simon (1987) formalizes the view of continuous time as "discrete, but with an arbitrarily fine grid" for deterministic timing games by restricting a given strategy profile to ever finer grids and identifying a limit outcome. Steg (2016) shows that Fudenberg and Tirole's (1985) preemption equilibrium is the limit of subgame-perfect equilibria of discrete-time approximations of the game.

[8] See Steg and Thijssen (2015) for a real-option game with both first- and second-mover advantages arising dynamically and in which exercise occurs in both regimes in a Markov perfect equilibrium in mixed strategies. Steg (2015b) uses the present framework to construct and analyze subgame-perfect equilibria in mixed strategies for general symmetric timing games.

second-mover advantages or payoff monotonicities. Laraki et al. (2005) consider general deterministic N-player games with payoffs that are just continuous functions of time (for given identities of first-movers). They prove that ε-equilibria always exist but that exact equilibria do not necessarily exist.

On the other hand, as we emphasize uncertainty, the literature on Dynkin games with a large tradition in mathematics also warrants mentioning here. Classically, however, these are two-person, *zero-sum* timing games, and the central question is the existence of a *value* under varying conditions, respectively whether there is a saddle point in strategy space. Here, we just refer to Touzi and Vieille's (2002) more recent work as their payoff processes are very general and as they use a different concept of mixed strategies (but without considering subgames). Touzi and Vieille (2002) prove that many more Dynkin games have a value if the players are allowed to randomize over stopping times and that such strategies are payoff-equivalent to state-dependent cumulative distributions functions as used here. Quite recently, the two strands of the literature have begun to merge by considering stochastic timing games with non-zero-sum payoffs. Hamadène and Zhang (2010), for instance, have proven the existence of Nash equilibrium for 2-player games with a general second-mover advantage.[9]

The type of application on which we focus is strategic investment under uncertainty. Some early models that we will revisit include those by Weeds (2002) and similar models by Pawlina and Kort (2006), Mason and Weeds (2010), and followers. We propose strategies that support the equilibrium outcomes described in these papers.

1.2. Organization of the paper

We begin by defining the stochastic timing game and our notion of subgames and mixed strategies in Section 2. In Section 3 we derive equilibria in extended mixed strategies for preemption games with local first-mover advantages. Our concepts and general results are illustrated by several applications in Section 4. In Section 5 we discuss how our framework can be applied to models that qualitatively extend the scope of the literature on timing games, including games with more than two players. The Appendix presents all proofs and some technical results.

2. A framework for stochastic timing games

We consider a timing game between two players in continuous time $t \in \mathbb{R}_+$. There may be uncertainty about a relevant state of the world, and exogenously evolving information about that state, which are represented by a fixed filtered probability space $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P\big)$.

[9] See also Hamadène and Hassani (2014) for an extension to N players using a similar approach. Laraki and Solan (2013) make less assumptions concerning the incentives in a 2-player game. Consequently, even allowing for mixed strategies, they can only prove existence of ε-equilibria. None of these papers considers subgame perfection and their constructed equilibria will generally not be subgame-perfect.

Each player can move once. Moves are irreversible, so any move changes the *mode* of the game that summarizes who has already moved. A change of the mode is instantaneously observable. If both players move simultaneously in the initial mode, the game ends. If only one player moves, the game is reduced to a simple decision problem for the remaining player. By the change of the mode, the remaining player gets a second chance to move in the same instant, or at any other point of remaining time. We focus on modeling the initial mode of the game in which nobody has moved, and encode the behavior of any remaining player in the (continuation) payoffs at the time of the first move, resp. the first change of mode.

As explained in the Introduction, strategies are open loop concerning other players' moves within each mode. They are plans when to move (or "stop") if no one else moves before. Strategies may, however, condition on the dynamic information about the state. Feasible state-dependent plans are thus *stopping times* $\tau \colon \Omega \to [0, \infty]$, such that for every $t \in \mathbb{R}_+$ the event "stop before time t" is identifiable by the available information about the state, i.e., $\{\tau \leq t\} := \{\omega \in \Omega \,|\, \tau(\omega) \leq t\} \in \mathscr{F}_t$. The value $\tau(\omega) = \infty$ is interpreted as never stopping. Denote the set of all stopping times by $\mathscr{T}$, which are the *pure strategies* for the initial mode of the game.

If players $i, j \in \{1, 2\}$, $i \neq j$, plan to stop at the stopping times τ_i and τ_j, respectively, then the initial mode of the game ends at the stopping time $\min(\tau_i, \tau_j)$. Player i is called the *leader* if $\tau_i < \tau_j$, and the *follower* if $\tau_i > \tau_j$. Otherwise, the players move simultaneously at $\tau_i = \tau_j$. Taking the (optimal, mode-contingent) behavior of any player who becomes follower at an arbitrary stopping time $\tau \in \mathscr{T}$ as given, let L^i_τ and F^i_τ denote the implied expected continuation payoffs for player i as leader or follower, respectively. Let M^i_τ denote the expected payoff at τ if the players stop simultaneously. The indices i allow for the role-specific payoffs to depend also on the identities of the players.

Then the expected payoff to player i at time 0 is

$$\pi_i\left(\tau_i, \tau_j\right) := E\big[L^i_{\tau_i} \mathbf{1}_{\{\tau_i < \tau_j\}} + F^i_{\tau_j} \mathbf{1}_{\{\tau_j < \tau_i\}} + M^i_{\tau_i} \mathbf{1}_{\{\tau_i = \tau_j\}}\big]. \tag{2.1}$$

We assume that the continuation payoffs at the time of the first move are given by six stochastic processes $L^i = (L^i_t)_{t \geq 0}$, $F^i = (F^i_t)_{t \geq 0}$ and $M^i = (M^i_t)_{t \geq 0}$ that are evaluated at the relevant stopping time. These processes are our basic data for the game and we need to make some standard regularity assumptions in order to have well defined (stopping) problems in the following.

Assumption 2.1.

(1) *The filtration $\mathbf{F} := (\mathscr{F}_t)_{t \geq 0}$ satisfies the usual conditions: it is right-continuous and* complete *(i.e., every $\mathscr{F}_t$ contains all subsets of any P-null set from $\mathscr{F}$).*

(2) *The processes L^i, F^i and M^i, $i \in \{1, 2\}$, are adapted, right-continuous (a.s.) and of class* (D)*, each M^i having an ($\mathscr{F}$-measurable) extension to $t = \infty$ with $E[|M^i_\infty|] < \infty$.*

Remark 2.2.

(1) A measurable process X is of class (D) if the family $\{X_\tau \,|\, \tau \in \mathscr{T}, \tau < \infty\}$ is uniformly integrable, so that the family is bounded in $L^1(P)$ and pointwise convergence of X at a stopping time implies convergence in $L^1(P)$ as well. This is a mild regularity condition implied by, e.g., either $E[\sup_t |X_t|] < \infty$ or $\sup_\tau E[|X_\tau|^p] < \infty$ for some $p > 1$. We may equivalently define any extension $X_\infty \in L^1(P)$ and consider *all* stopping times (possibly taking the value ∞) in the previous set; cf. Lemma B.1 in the appendix.

(2) It depends on the setting whether there is a natural payoff if both players "never stop", which may be some limit of M^i or of L^i. In any case we denote the payoff by M^i_∞ for a convenient notation. For convenience we also define
$$F^i_\infty := M^i_\infty.$$

(3) By assumption, L^i_τ and F^i_τ give the values implied by a potential follower problem at any *stopping time* τ. If the follower problems are to be modeled explicitly, one needs to take care that there exist processes $(L^i_t)_{t\geq 0}$ and $(F^i_t)_{t\geq 0}$ with the assumed properties; see Example 2.5 below or, for more general conditions, Steg (2015a).

Now we can give a first definition that can be seen as the *normal form* of (the initial mode of) a timing game.

Definition 2.3. A timing game Γ is a tuple
$$\Big(\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P\big), \mathscr{T} \times \mathscr{T}, \big(L^i, F^i, M^i\big)_{i=1,2}, (\pi_i)_{i=1,2}\Big)$$
consisting of a filtered probability space $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P\big)$, stopping times $\mathscr{T}$ as pure strategies, adapted processes $\big(L^i, F^i, M^i\big)_{i=1,2}$ satisfying Assumption 2.1, and payoffs π_i given by (2.1).

The dynamic view on the game will consist in reconsidering the normal form at any time at which a move could occur, resp. at which some player could stop, and requiring the sets of open loop strategies for all subgames to be consistent. Before, however, we present two examples that serve to illustrate our framework and why we model the concepts as we do; they will be analyzed in detail in Section 4.

Example 2.4. *A stochastic grab-the-dollar-game.* This stochastic variant of the grab-the-dollar game presented in Fudenberg and Tirole (1985) will help to motivate our formulation of mixed strategies and illustrate the role of dynamic information about the state.

We consider an American and her European friend playing the grab-the-dollar game. When the European wins the dollar, he has to turn it into Euros, at a stochastic exchange rate given by an adapted, strictly positive and right-continuous process X. If both players grab the dollar at the same time, they pay a penalty of 1 currency unit in their local currency.

Let $0 < \exp(-r) < 1$ be the discount factor for both players. The payoffs are thus $L_t^1 = \exp(-rt)$ for the American and $L_t^2 = X_t \exp(-rt)$ for the European if they win the dollar at time $t \in \mathbb{R}_+$, respectively. As in the usual grab-the-dollar game, $F_t^1 = F_t^2 = 0$, and for the simultaneous stopping payoffs we set $M_t^1 = M_t^2 = -\exp(-rt)$ and $M_\infty^1 = M_\infty^2 = 0$.

Example 2.5. *Preemptive market entry.* This is a basic model of preemptive investment under uncertainty, which will serve to illustrate how to apply our framework to similar models in the literature, to overcome evident issues in their analysis. The model corresponds to, e.g., that of Grenadier (1996) without construction delay, that of Weeds (2002) without initial R&D and that of Pawlina and Kort (2006) with symmetric investment cost.

Two firms $i = 1, 2$ have the opportunity to invest irreversibly in the same market. The return from investment, in particular the profit flow from operating in the market, is uncertain. Assume the duopoly profit flow when both firms have invested is given by the observable, non-negative process X (a geometric Brownian motion for concreteness). If only one firm is present in the market, it can realize a monopoly markup and increase the profit flow to MX for some constant $M > 1$. There is a sunk cost of investment $I > 0$. We assume that profits are discounted at a common and constant rate $r > 0$.

If some firm invests at time $\tau \in \mathscr{T}$ as the leader, the other firm that becomes follower obtains the option to (re-)determine an optimal investment time in reaction. The follower's payoff is thus the value function of the optimal stopping problem

$$F_\tau^i := \sup_{\tau' \geq \tau} E\biggl[\int_{\tau'}^\infty e^{-rs}(X_s - rI)\,ds \,\bigg|\, \mathscr{F}_\tau\biggr].$$

Immediate investment is still feasible – assuming zero reaction lags in continuous time. Suppose the follower's problem has a solution. Denote it by $\tau^F(\tau) \in \mathscr{T}$ to indicate the dependence on τ. Then the leader enjoys the monopoly profit only on $[\tau, \tau^F(\tau))$, and the leader's payoff is thus given by

$$L_\tau^i := E\biggl[\int_\tau^{\tau^F(\tau)} e^{-rs}(MX_s - rI)\,ds + \int_{\tau^F(\tau)}^\infty e^{-rs}(X_s - rI)\,ds \,\bigg|\, \mathscr{F}_\tau\biggr].$$

The payoff from simultaneous investment at $\tau \in \mathscr{T}$ is simply

$$M_\tau^i := E\biggl[\int_\tau^\infty e^{-rs}(X_s - rI)\,ds \,\bigg|\, \mathscr{F}_\tau\biggr].$$

Given sufficient regularity of the process X, such as continuity and the strong Markov property, there are indeed right-continuous, adapted processes $(L_t^i)_{t\geq 0}$, $(F_t^i)_{t\geq 0}$ and $(M_t^i)_{t\geq 0}$ that, if evaluated at any $\tau \in \mathscr{T}$, yield the previously specified payoffs (see Section 4.2).

2.1. Subgames

The strategy spaces $\mathscr{T}$ alone would make the analysis of the game essentially static. For a truly dynamic view on the game, and for ruling out non-credible threats, one also has to consider behavior off the intended path of play, i.e., subgames and corresponding strategies. Focussing on the initial mode of the game, we consider hypothetical situations in which nobody has moved, but in which someone could move. The feasible state-dependent dates to schedule moves are stopping times. We thus consider stopping times also as dates at which strategies can be revisited.

In discrete time one can characterize a stopping time τ and the information about the state available at τ by the events $\{\omega \in \Omega \,|\, \tau(\omega) = t\} \in \mathscr{F}_t$ for the different periods t. In continuous time, however, it is not sufficient to consider deterministic times $t \in \mathbb{R}_+$ and their distinguishable events $\mathscr{F}_t$ (as it is typically done in the literature). Stopping times in continuous time generate a much richer dynamic information structure, the σ-fields $\mathscr{F}_\tau$, that have to be addressed directly.[10]

The problems arising in continuous time can be seen by means of one of the most prominent examples for a stopping time: the first moment a Brownian motion exceeds a certain fixed value $b > 0$. This "first passage time" (or hitting time) has a continuous distribution with full support on $(0, \infty)$.[11] If we wanted to infer behavior at such a hitting time τ from plans for deterministic times t, i.e., $\mathscr{F}_t$-measurable objects, and tried to construct a state-dependent plan via $\tau(\omega) = t$, this would mean aggregating uncountably many events of probability zero and would not give a measurable object in general.

Therefore we reconsider the game at every stopping time and use stopping times to characterize the starts of arbitrary subgames in which nobody has moved. We take again the normal form view on each subgame. The dynamic perspective on the full game will then be composed of considering all subgames in a consistent way.

Definition 2.6. Let $\vartheta \in \mathscr{T}$ be a stopping time. Let Γ be a timing game. The subgame Γ^ϑ starting at the stopping time ϑ is the tuple

$$\Big(\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P\big), \mathscr{T}_\vartheta \times \mathscr{T}_\vartheta, \big(L^i, F^i, M^i\big)_{i=1,2}, \big(\pi_i^\vartheta\big)_{i=1,2}\Big)$$

consisting of the filtered probability space $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P\big)$, stopping times greater equal ϑ

$$\mathscr{T}_\vartheta := \{\tau \in \mathscr{T} \,|\, \tau(\omega) \geq \vartheta(\omega)\ \forall \omega \in \Omega\}$$

[10] For any stopping time $\tau \in \mathscr{T}$, the σ-field $\mathscr{F}_\tau$ is defined as $\{A \in \mathscr{F}_\infty \,|\, A \cap \{\tau \leq t\} \subset \mathscr{F}_t \forall t \in \mathbb{R}_+\}$ (where $\mathscr{F}_\infty$ is the σ-field generated by $\bigcup_{t\in\mathbb{R}_+} \mathscr{F}_t$). The σ-fields $\mathscr{F}_\tau$ provide a "more general concept of a past" than $\mathscr{F}_t$ for deterministic times $t \in \mathbb{R}_+$ (Revuz and Yor, 1999).

[11] The first passage time has the density $b(2\pi t^3)^{-\frac{1}{2}} e^{-b^2/2t} > 0$ on $(0, \infty)$; see, e.g., Revuz and Yor (1999), Section II.3.

as pure strategies, adapted processes $\left(L^i, F^i, M^i\right)_{i=1,2}$ and conditional payoffs

$$\pi_i^\vartheta\left(\tau_i, \tau_j\right) = E\left[L^i_{\tau_i}\mathbf{1}_{\{\tau_i<\tau_j\}} + F^i_{\tau_j}\mathbf{1}_{\{\tau_j<\tau_i\}} + M^i_{\tau_i}\mathbf{1}_{\{\tau_i=\tau_j\}} \,\middle|\, \mathscr{F}_\vartheta\right], \quad \tau_i, \tau_j \in \mathscr{T}_\vartheta.$$

Our definition of subgames can also be seen as an analogy to the general approach to stopping problems for a single decision maker. The solution of an optimal stopping problem can be represented as a consistent collection of contingent plans for starting from *any stopping time*, representing the dynamic programming principle in continuous time.[12] Similarly, our approach allows us to speak meaningfully of contingent plans for subgames and to define subgame-perfect equilibria.

2.1.1. Mixed strategies in subgames

Many timing games have no equilibria in pure strategies. We thus introduce mixed strategies, following the approach by Fudenberg and Tirole (1985), but generalizing it to our stochastic setting. These strategies are first formulated for every subgame, indexed by the start $\vartheta \in \mathscr{T}$. Then we will impose natural consistency conditions on strategies for different subgames, implying that our approach becomes equivalent with behavior strategies when applied to discrete time.

The starting point for a mixed strategy is a distribution function G_i^ϑ over remaining time.[13] G_i^ϑ may depend on the state of the world in the sense that the cumulative probability of having stopped up to any future time t must be inferable from the state information provided by the filtration, $\mathscr{F}_t$.

Distributions over time are still not sufficient to obtain any equilibria in some very basic, well-behaved games with preemption incentives, which are central for applications.[14] Therefore we also generalize the strategy extensions α_i^ϑ of Fudenberg and Tirole (1985) as a coordination device for stochastic models. These extensions will specify time-dependent *conditional* stopping probabilities in continuous time, i.e., not *cumulative* probabilities. They should thus not charge the past $[0, \vartheta)$, either, but need not be monotone.

Definition 2.7. Fix a stopping time $\vartheta \in \mathscr{T}$. An *extended mixed strategy* for player $i \in \{1, 2\}$ for the subgame Γ^ϑ starting at ϑ, also called *ϑ-strategy*, is a pair of processes $\left(G_i^\vartheta, \alpha_i^\vartheta\right)$ taking values in $[0, 1]$, respectively, with the following properties.

(1) G_i^ϑ is adapted. It is a.s. right-continuous, non-decreasing, and satisfying $G_i^\vartheta(s) = 0$ for all $s < \vartheta$.

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[12] See, e.g., El Karoui (1981) for the general theory of optimal stopping and the concept of the *Snell envelope*.

[13] An alternative approach is to randomize over stopping times before the game starts. Touzi and Vieille (2002) show that the two approaches are payoff-equivalent. They do not consider, however, any notion of subgame (perfection) or further extensions as we do.

[14] See, e.g., Hendricks and Wilson (1992) for a deterministic, arbitrarily smooth preemption game without equilibrium in mixed strategies in the form of distributions over time.

(2) α_i^ϑ is progressively measurable.[15] It is a.s. right-continuous in all $t \in \mathbb{R}_+$ for which $\alpha_i^\vartheta(t) \in (0,1)$ and satisfying $\alpha_i^\vartheta(s) = 0$ for all $s < \vartheta$.

(3)
$$\alpha_i^\vartheta(t) > 0 \Rightarrow G_i^\vartheta(t) = 1 \quad \text{for all } t \geq 0 \quad \text{a.s.}$$

We further define $G_i^\vartheta(0-) \equiv 0$, $G_i^\vartheta(\infty) \equiv 1$ and $\alpha_i^\vartheta(\infty) \equiv 1$ for every extended mixed strategy.

Remark 2.8.

(1) As in Fudenberg and Tirole (1985), the extensions α_i^ϑ are a coordination instrument when the distribution functions G_i^ϑ jump to 1, aiming to capture richer limit outcomes from discrete-time approximations of the game. When some α_i^ϑ is positive, some move will happen instantaneously, and the relative values of α_1^ϑ and α_2^ϑ will determine the probabilities of each player becoming leader or stopping simultaneously with the other. Right-continuity of the processes α_i^ϑ is important to determine limit outcomes. We do not require it when $\alpha_i^\vartheta(t)$ is zero or one, in order to deal with some important classes of games (see Remark 3.2 and Section 4 below).

(2) Progressive measurability ensures enough structure in the time domain such that $\alpha_i^\vartheta(\tau)$ will be $\mathscr{F}_\tau$-measurable for any $\tau \in \mathscr{T}$, e.g., for the crucial $\tau = \inf\{t \geq 0 \,|\, \alpha_i^\vartheta(t) > 0\}$. $G_i^\vartheta(0-) \equiv 0$ and the terminal values for $t = \infty$ are defined for notational convenience in the definition of payoffs.

Our analysis of Examples 2.4 and 2.5 in Section 4 below will illustrate how strategies need to react to the dynamic information about the state of the world.

2.1.2. Outcomes and payoffs

We have to generalize the payoffs (2.1) from the start of the game to subgames and (extended) mixed strategies. Therefore we first determine outcome probabilities – of who stops first and when – that result from a pair of extended mixed strategies $\big(G_1^\vartheta, \alpha_1^\vartheta\big)$ and $\big(G_2^\vartheta, \alpha_2^\vartheta\big)$ for a given subgame starting at $\vartheta \in \mathscr{T}$. The outcome distribution will then be mapped to expected payoffs via the payoff processes.

The mixed strategy profile $(G_1^\vartheta, G_2^\vartheta)$ alone will be rather standard to handle, so we concentrate on discussing the extensions $(\alpha_1^\vartheta, \alpha_2^\vartheta)$. The intuition taken from Fudenberg and Tirole (1985) is to enable the players to put a positive stopping probability (conditional on nobody having stopped before) on every

[15] For every $t \in \mathbb{R}_+$, the function $(\omega, s) \mapsto \alpha_i^\vartheta(\omega, s)$ must be $\mathscr{F}_t \otimes \mathscr{B}([0,t])$-measurable on the restricted domain $\Omega \times [0,t]$ (where $\mathscr{B}([0,t])$ denotes the Borel sets of $[0,t]$). This is stronger than (i.e., it implies) adaptedness, but weaker than (i.e., implied by) optionality, which holds for G_i^ϑ by right-continuity. However, we can also assume α_i^ϑ to be optional without loss by using its unique optional projection ${}^o\alpha_i^\vartheta$. Indeed, at any $\tau \in \mathscr{T}$ the two agree a.s. by definition and ${}^o\alpha_i^\vartheta$ is right-continuous where α_i^ϑ is so by Lemma B.3, such that the outcome probabilities in Definition 2.9 below remain the same.

time $t \geq \vartheta$, denoted by $\alpha_i^\vartheta(t)$, mimicking a discrete-time game on an "infinitely fine" grid, or representing the limit of a sequence of strategies for discrete-time games with vanishing period length.

The idea can be illustrated by means of the grab-the-dollar game; we use our stochastic variant, Example 2.4. On any discrete grid of time, this game has several subgame-perfect equilibria with some player grabbing the dollar with probability one in every given period and the opponent abstaining, so there is perfect coordination to avoid costly collisions. The one who grabs can be, e.g., always player 1, because she has no incentive to wait, or both players can grab alternately.[16] In addition, there is a fully mixed subgame-perfect equilibrium: If player 1 grabs with probability $L^2_{t_n}/(L^2_{t_n} - M^2_{t_n}) = X_{t_n}/(X_{t_n}+1)$ in every period t_n (recall that the exchange rate X is strictly positive) and player 2 with probability $L^1_{t_n}/(L^1_{t_n} - M^1_{t_n}) = 1/2$, then every (pure or mixed) strategy gives each player the payoff 0. As the period length vanishes, the limit of each strategy is to grab immediately. Nevertheless, the probability of a collision does not converge to one, but $X_0/(2X_0+1)$.

More generally, limit outcomes are easy to identify if the families $(\alpha_1^\vartheta(t), \alpha_2^\vartheta(t))_{t\geq\vartheta}$ are sufficiently regular with respect to time. Where $\alpha_i^\vartheta(\cdot)$ is positive and right-continuous, there are in the limit arbitrarily many consecutive stopping probabilities that vary arbitrarily little in arbitrarily short time intervals. Thus, in the limit, stopping occurs immediately with probability one (reflected in $G_i^\vartheta(\cdot) = 1$), and the limit distribution over the identity of who stops first – player 1, player 2, or both – is that of an infinitely repeated game with constant stage stopping probabilities. The distribution is given by the functions μ_L and μ_M from $[0,1]^2 \setminus \{(0,0)\}$ to $[0,1]$ defined by

$$\mu_L(x,y) := x(1-y)\sum_{n=0}^{\infty}[(1-x)(1-y)]^n = \frac{x(1-y)}{x+y-xy}$$

and
$$\mu_M(x,y) := \frac{xy}{x+y-xy}.$$

$\mu_L(a_i,a_j)$ is the probability that player i stops first if players i and j stop with probabilities a_i and a_j, respectively, in every stage of a repeated game. $\mu_M(a_i,a_j)$ is the probability of simultaneous stopping and $1-\mu_L(a_i,a_j)-\mu_M(a_i,a_j) = \mu_L(a_j,a_i)$ is that of player j stopping first.

The limit outcome can incorporate special "first round" behavior. If only one player uses the extension $\alpha_i^\vartheta(t) > 0$, whereas the other uses a distribution G_j^ϑ with an isolated atom, i.e., $\Delta G_j^\vartheta(t) := G_j^\vartheta(t) - G_j^\vartheta(t-) > 0$ and $\alpha_j^\vartheta(\cdot) = 0$ on an interval from t, the outcome is that of stopping with (conditional) probabilities $\big(\alpha_i^\vartheta(t), \Delta G_j^\vartheta(t)/[1-G_j^\vartheta(t-)]\big)$ in the first round and with $\big(\alpha_i^\vartheta(t), 0\big)$ thereafter. As a result, player i becomes leader at t unless j stops in the first round. This case extends to α_i^ϑ having positive values arbitrarily close to t from the right but

[16] It may or not be an equilibrium that player 2 always grabs and player 1 never, because the payoff $X_t \exp(-rt)$ may be increasing in expectation.

$\alpha_i^\vartheta(t) = 0$, because if $\alpha_j^\vartheta(\cdot) = 0$ to the right of t, player i becomes leader once t is passed due to the arbitrarily many positive consecutive stopping probabilities before the next possible atom of G_j^ϑ. Thus, we do not need right-continuity to identify the limit outcomes in these cases, and neither so if $\alpha_i^\vartheta(t) = 1$, when player i stops definitely in the first round.

Difficulties arise if both α_i^ϑ start to be positive at the same time and at least one still equals zero at that point. One problem is that μ_L has no continuous extension at the origin (whereas that of μ_M is $\mu_M(0,0) := 0$), and another is the possible lack of right-continuity of the extensions. Fudenberg and Tirole (1985) require differentiability with a positive derivative in such a case, which is not feasible for the (asymmetric and/or stochastic) models we are going to deal with. Instead, we take a right-hand limit of the outcome distribution, again combined with a "first round" if some α_i^ϑ is not right-continuous (and thus equals zero at the critical point).

The following outcome probabilities generated by the extensions, of who stops first at

$$\hat\tau^\vartheta := \inf\{t \geq \vartheta \,|\, \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\},$$

are defined as seen from ϑ, so their leading term is the probability of reaching $\hat\tau^\vartheta$ with no player having stopped before, $\big(1 - G_1^\vartheta(\hat\tau^\vartheta-)\big)\big(1 - G_2^\vartheta(\hat\tau^\vartheta-)\big)$. Note that Definition 2.7 (iii) implies $\big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big) = \Delta G_i^\vartheta(\hat\tau^\vartheta)$ if $\hat\tau^\vartheta = \inf\{t \geq \vartheta \,|\, \alpha_i^\vartheta(t) > 0\}$. If $\big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big) = 0$, we define $\big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big)/\big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big) := 0$ for notational convenience.

Definition 2.9. Given $\vartheta \in \mathscr{T}$ and a pair of extended mixed strategies $\big(G_1^\vartheta, \alpha_1^\vartheta\big)$, $\big(G_2^\vartheta, \alpha_2^\vartheta\big)$, the *outcome probabilities* $\lambda_{L,1}^\vartheta$, $\lambda_{L,2}^\vartheta$ and λ_M^ϑ for player 1 becoming leader, player 2 becoming leader and simultaneous stopping, respectively, at $\hat\tau^\vartheta$ are defined as follows. Let $i, j \in \{1, 2\}$, $i \neq j$.
If $\hat\tau^\vartheta < \hat\tau_j^\vartheta := \inf\{t \geq \vartheta \,|\, \alpha_j^\vartheta(t) > 0\}$, then

$$\lambda_{L,i}^\vartheta := \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1 - G_j^\vartheta(\hat\tau^\vartheta-)\big)\left[1 - \frac{\Delta G_j^\vartheta(\hat\tau^\vartheta)}{1 - G_j^\vartheta(\hat\tau^\vartheta-)}\right] = \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1 - G_j^\vartheta(\hat\tau^\vartheta)\big),$$

$$\lambda_M^\vartheta := \Delta G_i^\vartheta(\hat\tau^\vartheta)\big(1 - G_j^\vartheta(\hat\tau^\vartheta-)\big)\alpha_i^\vartheta(\hat\tau^\vartheta)\frac{\Delta G_j^\vartheta(\hat\tau^\vartheta)}{1 - G_j^\vartheta(\hat\tau^\vartheta-)} = \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\alpha_i^\vartheta(\hat\tau^\vartheta).$$

If $\hat\tau^\vartheta < \hat\tau_i^\vartheta := \inf\{t \geq \vartheta \,|\, \alpha_i^\vartheta(t) > 0\}$, then

$$\lambda_{L,i}^\vartheta := \big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big)\Delta G_j^\vartheta(\hat\tau^\vartheta)\frac{\Delta G_i^\vartheta(\hat\tau^\vartheta)}{1 - G_i^\vartheta(\hat\tau^\vartheta-)}\big(1 - \alpha_j(\hat\tau^\vartheta)\big) = \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\big(1 - \alpha_j(\hat\tau^\vartheta)\big),$$

$$\lambda_M^\vartheta := \big(1 - G_i^\vartheta(\hat\tau^\vartheta-)\big)\Delta G_j^\vartheta(\hat\tau^\vartheta)\frac{\Delta G_i^\vartheta(\hat\tau^\vartheta)}{1 - G_i^\vartheta(\hat\tau^\vartheta-)}\alpha_j(\hat\tau^\vartheta) = \Delta G_i^\vartheta(\hat\tau^\vartheta)\Delta G_j^\vartheta(\hat\tau^\vartheta)\alpha_j^\vartheta(\hat\tau^\vartheta).$$

If $\hat\tau^\vartheta = \hat\tau_1^\vartheta = \hat\tau_2^\vartheta$ and either $\max\big\{\alpha_1^\vartheta(\hat\tau^\vartheta), \alpha_2^\vartheta(\hat\tau^\vartheta)\big\} = 1$ or $\min\big\{\alpha_1^\vartheta(\hat\tau^\vartheta), \alpha_2^\vartheta(\hat\tau^\vartheta)\big\} >$

0, then

$$\lambda^\vartheta_{L,i} := \Delta G^\vartheta_i(\hat\tau^\vartheta)\Delta G^\vartheta_j(\hat\tau^\vartheta)\mu_L(\alpha^\vartheta_i(\hat\tau^\vartheta),\alpha^\vartheta_j(\hat\tau^\vartheta)),$$

$$\lambda^\vartheta_{M} := \Delta G^\vartheta_i(\hat\tau^\vartheta)\Delta G^\vartheta_j(\hat\tau^\vartheta)\mu_M(\alpha^\vartheta_1(\hat\tau^\vartheta),\alpha^\vartheta_2(\hat\tau^\vartheta)).$$

If $\hat\tau^\vartheta = \hat\tau^\vartheta_1 = \hat\tau^\vartheta_2$, $\max\{\alpha^\vartheta_1(\hat\tau^\vartheta),\alpha^\vartheta_2(\hat\tau^\vartheta)\} < 1$ and $\min\{\alpha^\vartheta_1(\hat\tau^\vartheta),\alpha^\vartheta_2(\hat\tau^\vartheta)\} = 0$, then

$$\begin{aligned}\lambda^\vartheta_{L,i} := {} & \Delta G^\vartheta_i(\hat\tau^\vartheta)\Delta G^\vartheta_j(\hat\tau^\vartheta)\big(1-\alpha^\vartheta_j(\hat\tau^\vartheta)\big)\\ & \cdot\Bigg(\alpha^\vartheta_i(\hat\tau^\vartheta) + \big(1-\alpha^\vartheta_i(\hat\tau^\vartheta)\big)\frac{1}{2}\Big\{\liminf_{\substack{t\searrow\hat\tau^\vartheta\\ \alpha^\vartheta_i(t)+\alpha^\vartheta_j(t)>0}} \mu_L(\alpha^\vartheta_i(t),\alpha^\vartheta_j(t))\\ & \qquad + \limsup_{\substack{t\searrow\hat\tau^\vartheta\\ \alpha^\vartheta_i(t)+\alpha^\vartheta_j(t)>0}} \mu_L(\alpha^\vartheta_i(t),\alpha^\vartheta_j(t))\Big\}\Bigg),\end{aligned}$$

$$\begin{aligned}\lambda^\vartheta_{M} := {} & \Delta G^\vartheta_i(\hat\tau^\vartheta)\Delta G^\vartheta_j(\hat\tau^\vartheta) - \lambda^\vartheta_{L,i} - \lambda^\vartheta_{L,j}\\ = {} & \Delta G^\vartheta_i(\hat\tau^\vartheta)\Delta G^\vartheta_j(\hat\tau^\vartheta)\big(1-\alpha^\vartheta_i(\hat\tau^\vartheta)\big)\big(1-\alpha^\vartheta_j(\hat\tau^\vartheta)\big)\\ & \cdot\mu_M(\alpha^\vartheta_1(\hat\tau^\vartheta+),\alpha^\vartheta_2(\hat\tau^\vartheta+)) \qquad \text{if } \alpha^\vartheta_1(\hat\tau^\vartheta+) \text{ and } \alpha^\vartheta_2(\hat\tau^\vartheta+) \text{ exist.}\end{aligned}$$

Remark 2.10.

(1) $\lambda^\vartheta_{L,i}$ is also the probability of player j becoming follower at $\hat\tau^\vartheta$. In all cases it holds that $\lambda^\vartheta_M + \lambda^\vartheta_{L,i} + \lambda^\vartheta_{L,j} = \big(1 - G^\vartheta_i(\hat\tau^\vartheta-)\big)\big(1 - G^\vartheta_j(\hat\tau^\vartheta-)\big)$, the probability of reaching $\hat\tau^\vartheta$ with no player having stopped before. Dividing the outcome probabilities by their sum where feasible yields the corresponding conditional probabilities at $\hat\tau^\vartheta$, which will be required to satisfy time consistency.

(2) In the last case we might not have a (right-hand) limit of μ_L if $\alpha^\vartheta_1(\hat\tau^\vartheta) = \alpha^\vartheta_2(\hat\tau^\vartheta) = 0$ even when both $\alpha^\vartheta_\cdot$ are continuous, whereas a limit probability for simultaneous stopping exists as soon as $\alpha^\vartheta_1(\hat\tau^\vartheta+)$ and $\alpha^\vartheta_2(\hat\tau^\vartheta+)$ do because μ_M is continuous at $(0,0)$.[17]

Here we differ from Fudenberg and Tirole (1985), who ask for right-differentiability and a positive derivative to apply a Taylor expansion, which is a too strong requirement for asymmetric or stochastic models; cf. Section 3. Taking instead the symmetric combination of lim inf and lim sup ensures consistency whenever the limit exists, independence of the players' names, and that λ^ϑ_M coincides with its associated limit whenever

[17]If $\max\{\alpha^\vartheta_1(\hat\tau^\vartheta),\alpha^\vartheta_2(\hat\tau^\vartheta)\} < 1$ and $\alpha^\vartheta_1(\hat\tau^\vartheta+)$, $\alpha^\vartheta_2(\hat\tau^\vartheta+)$ exist, then the limit of μ_L can fail *only* if $\alpha^\vartheta_1(\hat\tau^\vartheta+) = \alpha^\vartheta_2(\hat\tau^\vartheta+) = 0$, i.e., if $\alpha^\vartheta_1(\hat\tau^\vartheta) = \alpha^\vartheta_2(\hat\tau^\vartheta) = 0$; if $\alpha^\vartheta_i(\hat\tau^\vartheta+) > 0$, the limit of μ_L is determined by continuity. If the limit in a potential equilibrium does not exist, both players will be indifferent about the roles; see Lemma C.2.

the latter exists. Furthermore, our solution provides no incentives for the players to create ambiguity about the limit of μ_L by their choice of strategies (i.e., to exploit the rule of the last case): in Section 3 we show that if any player uses the extension, then the other has a pure best reply.

(3) We stress that as in Fudenberg and Tirole (1985) the coordination device ensures consistency with *well-behaved* limits from discrete-time approximations of the game while yielding sufficient coordination to model preemption as an equilibrium; see Section 3. Regarding any formal passage to the limit from discrete-time approximations, there is a trade-off between the richness of sequences (of strategies or equilibria) to be captured and the determination of well-defined limit outcomes. It is easy to construct timing games with equilibria in which the players take arbitrary turns of stopping in every period, which have no meaningful limit (neither in outcomes, nor in payoffs).

(4) Finally, as a technical remark, there are no conditional expectations in the definition, which one might expect as we are taking limits of "future" outcome probabilities that are state-dependent. However, we have pointwise right-hand limits and boundedness where $\alpha_i^\vartheta \in (0,1)$ and thus convergence of expectations when we apply the limit argument. Even if the $\liminf(\cdot)$ and $\limsup(\cdot)$ in the last case differ, they are progressively measurable by Theorem IV.33 (c) in Dellacherie and Meyer (1978) when seen as processes and hence $\mathscr{F}_\tau$-measurable at any $\tau \in \mathscr{T}$.

Now we can combine the outcome distribution from the extensions with those from the "standard" mixed strategies and integrate the payoff processes.

Definition 2.11. Given two extended mixed strategies $(G_i^\vartheta, \alpha_i^\vartheta)$, $(G_j^\vartheta, \alpha_j^\vartheta)$, $i, j \in \{1,2\}$, $i \neq j$, the *payoff* of player i in the subgame starting at $\vartheta \in \mathscr{T}$ is

$$\begin{aligned} V_i^\vartheta\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big) := E\bigg[& \int_{[0,\hat\tau^\vartheta)} \big(1 - G_j^\vartheta(s)\big) L_s^i \, dG_i^\vartheta(s) \\ & + \int_{[0,\hat\tau^\vartheta)} \big(1 - G_i^\vartheta(s)\big) F_s^i \, dG_j^\vartheta(s) \\ & + \sum_{s \in [0,\hat\tau^\vartheta)} \Delta G_i^\vartheta(s) \Delta G_j^\vartheta(s) M_s^i \\ & + \lambda_{L,i}^\vartheta L_{\hat\tau^\vartheta}^i + \lambda_{L,j}^\vartheta F_{\hat\tau^\vartheta}^i + \lambda_M^\vartheta M_{\hat\tau^\vartheta}^i \,\bigg|\, \mathscr{F}_\vartheta \bigg]. \end{aligned}$$

Lemma B.2 in the appendix shows that the pathwise integrals (which include possible jumps of the *right-continuous* integrators at 0, since i can indeed become leader/follower from an initial jump of $G_i^\vartheta / G_j^\vartheta$, resp.) are well defined under Assumption 2.1 and that the payoffs are bounded in $L^1(P)$ – uniformly across all feasible strategies.

Remark 2.12.

(1) If the players do not use the extensions (i.e., $\alpha_1^\vartheta = \alpha_2^\vartheta \equiv 0$ on $\mathbb{R}_+$), then $\hat\tau^\vartheta = \infty$, $\lambda_{L,1}^\vartheta = \lambda_{L,2}^\vartheta = 0$, and $\lambda_M^\vartheta = \Delta G_1^\vartheta(\infty)\Delta G_2^\vartheta(\infty)$, and thus the payoffs are the same as in the analogous model with only mixed strategies G_1^ϑ, G_2^ϑ. In either case any mass points at $t = \infty$ contribute $(1 - G_i^\vartheta(\infty-))(1 - G_j^\vartheta(\infty-))M_\infty^i$ due to $G_i^\vartheta(\infty) = \alpha_i^\vartheta(\infty) = 1$.

(2) It is also possible to interpret (α_1, α_2) as an *endogenous sharing rule* in the spirit of Simon and Zame (1990) as follows. Instead of (M^1, M^2), one regards the (pointwise) convex hull of $\{(L^1, F^2), (F^1, L^2), (M^1, M^2)\}$ as relevant payoffs for simultaneous stopping. The actual payoffs are a selection from this set that depends on (α_1, α_2), and the latter are (an endogenous) part of the solution. At each $\hat\tau_i^\vartheta$ the selection corresponds to the probabilities $(\lambda_{L,1}^\vartheta, \lambda_{L,2}^\vartheta, \lambda_M^\vartheta)$ if one sets $G_1^\vartheta = G_2^\vartheta = (\mathbf{1}_{\{t \geq \hat\tau_i^\vartheta\}})$ in Definition 2.9 and on $\{\hat\tau_i^\vartheta > \hat\tau_j^\vartheta\}$ also $\alpha_j^\vartheta \equiv 0$. Everywhere else, it remains $(M_\cdot^1, M_\cdot^2)$. Given this selection, one considers *non-extended* mixed strategies $(G_1^\vartheta, G_2^\vartheta)$ that yield the modified payoffs with probability $\Delta G_1^\vartheta(\hat\tau_i^\vartheta) \cdot \Delta G_2^\vartheta(\hat\tau_i^\vartheta)$. Then any pair $\big(G_1^\vartheta, \alpha_1^\vartheta\big)$, $\big(G_2^\vartheta, \alpha_2^\vartheta\big)$ of extended mixed strategies yields the same payoffs, whether evaluated as such or as mixed strategies with sharing rule. Furthermore, keeping the sharing rule (α_1, α_2) fixed, unilateral deviations to pure strategies of the form $G_i^\vartheta = (\mathbf{1}_{\{t \geq \tau\}})$ (that then typically do not satisfy $\alpha_i^\vartheta(t) > 0 \Rightarrow G_i^\vartheta(t) = 1$ anymore) yield payoffs equal to those from some extended mixed strategies satisfying $G_i^\vartheta = (\mathbf{1}_{\{t \geq \tau\}})$. Thus, by linearity there cannot be profitable deviations from any equilibrium in extended mixed strategies for the sharing rule (α_1, α_2), and these are also equilibria with endogenous sharing rule.

2.1.3. Time consistency and subgame-perfect equilibrium

Now the ϑ-strategies for all subgames can be combined to a strategy for the whole game to generate the dynamic perspective. Therefore, strategies for different subgames are required to be time consistent in the sense that Bayes' law holds, resp. that conditional stopping probabilities agree across strategies when possible (cf. Fudenberg and Tirole, 1985).

Definition 2.13. A *time-consistent extended mixed strategy* for player $i \in \{1, 2\}$ for the timing game Γ is a family of ϑ-strategies $\big(G_i, \alpha_i\big) = \big(G_i^\vartheta, \alpha_i^\vartheta\big)_{\vartheta \in \mathscr{T}}$ such that for all $\vartheta, \vartheta', \tau \in \mathscr{T}$ with $\vartheta \leq \vartheta' \leq \tau$ a.s. it holds that

$$G_i^\vartheta(t) = G_i^\vartheta(\vartheta'-) + \big(1 - G_i^\vartheta(\vartheta'-)\big) G_i^{\vartheta'}(t) \quad \text{for all } t \geq \vartheta' \quad \text{a.s.}$$

and

$$\alpha_i^\vartheta(\tau) = \alpha_i^{\vartheta'}(\tau) \quad \text{a.s.}$$

We denote the set of such strategies for each player by $\mathscr{S}$.

Note that there is in general a high level of redundancy in a family $\big(G_i^\vartheta, \alpha_i^\vartheta\big)_{\vartheta \in \mathscr{T}}$ because many stopping times coincide with a positive probability, so subgames may differ only on small events. Time-consistent strategies give unique and well-defined prescriptions in those cases, because they imply that $G_i^\vartheta = G_i^{\vartheta'}$ and $\alpha_i^\vartheta = \alpha_i^{\vartheta'}$ a.s. on the event $\{\vartheta = \vartheta'\}$ for all stopping times $\vartheta, \vartheta' \in \mathscr{T}$.[18]

The equilibrium concept is then natural.

Definition 2.14. A *subgame-perfect equilibrium* for the timing game is a pair $\big(G_1, \alpha_1\big)$, $\big(G_2, \alpha_2\big)$ of time-consistent extended mixed strategies such that for all $\vartheta \in \mathscr{T}$, $i, j \in \{1, 2\}$, $i \neq j$, and extended mixed strategies $\big(G_a^\vartheta, \alpha_a^\vartheta\big)$

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \geq V_i^\vartheta(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \quad \text{a.s.,}$$

i.e., such that every pair $\big(G_1^\vartheta, \alpha_1^\vartheta\big)$, $\big(G_2^\vartheta, \alpha_2^\vartheta\big)$ is an *equilibrium* in the subgame starting at $\vartheta \in \mathscr{T}$, respectively.

3. Preemption games

In this section we establish a quite general explicit form for subgame-perfect equilibria in games with preemption incentives. Our Proposition 3.1 shows how to model mutual preemption in continuous time with our strategy extensions. It resolves conceptual problems from which some of the proposed "equilibria" in the literature suffer and it can be applied to quite general models. Theorem 3.3 shows how to obtain subgame-perfect equilibria based on preemption in a general setting, whereas more specific applications will be presented in Section 4.

We begin by analyzing the payoffs that can result from extended mixed strategies, to show that equilibrium conditions have strong implications for the relevant choices of α_i^ϑ. Throughout this section we assume $F^i \geq M^i$ (a.s., for any $t \in \mathbb{R}_+$).[19]

Whenever some α_j^ϑ is positive, the (initial mode of the) game definitely ends, and by α_i^ϑ, player i can control the probabilities of becoming leader, becoming follower, or stopping simultaneously with player j, to an extent depending on α_j^ϑ. More specifically, suppose $\vartheta = \hat{\tau}_j^\vartheta = \inf\{t \geq \vartheta \,|\, \alpha_j^\vartheta(t) > 0\}$. Then $G_j^\vartheta(\vartheta) = 1$ and player i's payoff from any strategy $\big(G_i^\vartheta, \alpha_i^\vartheta\big)$ will be at most $V_i^\vartheta\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big) \leq \max\{F_\vartheta^i, \alpha_j^\vartheta(\vartheta) M_\vartheta^i + \big(1 - \alpha_j^\vartheta(\vartheta)\big) L_\vartheta^i\}$.[20] These payoffs result from stopping never or immediately, respectively, i.e., from G_i^ϑ putting

[18] Definition 2.13 implies the claimed property via $G_i^\vartheta = G_i^{\vartheta\wedge\vartheta'}$ on $\{\vartheta = \vartheta \wedge \vartheta'\}$ and $G_i^{\vartheta'} = G_i^{\vartheta\wedge\vartheta'}$ on $\{\vartheta' = \vartheta \wedge \vartheta'\}$; similarly for $\alpha_i^\cdot$.

[19] The assumption $F^i \geq M^i$ is very natural if the follower still has a stopping decision to make, such that F^i is the corresponding value function and simultaneous stopping is one option for the "follower". On a more abstract level the condition means that we are focussing on competitive models without a strict benefit from joint moves.

[20] This is easy to check, except for when the outcome probabilities $\lambda_{L,i}^\vartheta$ and λ_M^ϑ involve non-trivial limits due to $\vartheta = \hat{\tau}_1^\vartheta = \hat{\tau}_2^\vartheta$, $\min\{\alpha_1^\vartheta(\vartheta), \alpha_2^\vartheta(\vartheta)\} = 0$ and $\max\{\alpha_1^\vartheta(\vartheta), \alpha_2^\vartheta(\vartheta)\} < 1$; the verification for this case is given by Lemma C.1 in the appendix. The payoff is in fact a convex combination of F_ϑ^i and $\alpha_j^\vartheta(\vartheta) M_\vartheta^i + (1 - \alpha_j^\vartheta(\vartheta)) L_\vartheta^i$, possibly except for the case that $\alpha_j^\vartheta(\vartheta) = 0$ and $\lambda_M^\vartheta > 0$.

full mass on $\{\infty\}$ or $\{\vartheta\}$ (and $\alpha_i^\vartheta(t) = 0$ for all $t \in \mathbb{R}_+$). There is, thus, a best reply for player i that is pure.[21] Stopping (waiting) is strictly optimal iff

$$L^i_\vartheta - F^i_\vartheta > (<)\, \alpha^\vartheta_j(\vartheta)\big(L^i_\vartheta - M^i_\vartheta\big). \tag{3.1}$$

As $M^i_\vartheta \leq F^i_\vartheta$, waiting is optimal whenever $L^i_\vartheta \leq F^i_\vartheta$, and i can only be indifferent in that case if $\alpha^\vartheta_j(\vartheta)\big(F^i_\vartheta - M^i_\vartheta\big) = 0$, with $\alpha^\vartheta_j(\vartheta) = 1$ if $L^i_\vartheta < F^i_\vartheta$.

If $L^i_\vartheta > F^i_\vartheta$, i is indifferent iff

$$\alpha^\vartheta_j(\vartheta) = \frac{L^i_\vartheta - F^i_\vartheta}{L^i_\vartheta - M^i_\vartheta}, \tag{3.2}$$

which is in $(0, 1]$ a.s.

Based on (3.2) we can find equilibria where both players try to stop whenever both have a first-mover advantage, and where any hesitation by one player makes the other become leader; they can be interpreted as preemption in the region $\{(L^1 - F^1) \wedge (L^2 - F^2) > 0\}$. In order to prepare for subgame-perfect equilibria in asymmetric games where the preemption region is not reached immediately (e.g., Theorem 3.3 below), we make sure that if one player is indifferent about becoming leader or follower, whereas the other has a strict preference, then the latter can realize the advantage.[22]

Proposition 3.1. *Assume $F^i_t \geq M^i_t$ for all $t \in \mathbb{R}_+$, a.s. ($i = 1, 2$). For any $\tau \in \mathscr{T}$ let*

$$\tau^{\mathcal{P}}(\tau) := \inf\{u \geq \tau \,|\, (L^1_u - F^1_u) \wedge (L^2_u - F^2_u) > 0\}$$

(for every $\omega \in \Omega$). If $\vartheta \in \mathscr{T}$ satisfies $\vartheta = \tau^{\mathcal{P}}(\vartheta)$ a.s., then $\big(G^\vartheta_1, \alpha^\vartheta_1\big)$, $\big(G^\vartheta_2, \alpha^\vartheta_2\big)$ given by

$$\alpha^\vartheta_i(t) = \begin{cases} 1 & \text{if} \quad t = \inf\{u \geq t \,|\, (L^1_u - F^1_u) \wedge (L^2_u - F^2_u) > 0\}, \\ & \qquad L^j_t = F^j_t \text{ and } \big(L^i_t > F^i_t \text{ or } F^j_t = M^j_t\big), \\ \mathbf{1}_{\{L^1_t > F^1_t\}} \mathbf{1}_{\{L^2_t > F^2_t\}} \dfrac{L^j_t - F^j_t}{L^j_t - M^j_t} & \text{else} \end{cases}$$

and $G^\vartheta_i(t) = \mathbf{1}_{\{t \geq \vartheta\}}$ for any $t \in [\vartheta, \infty)$, $i, j \in \{1, 2\}$, $i \neq j$, are an equilibrium in the subgame starting at ϑ.

The resulting payoffs are $V^\vartheta_i(G^\vartheta_i, \alpha^\vartheta_i, G^\vartheta_j, \alpha^\vartheta_j) = F^i_\vartheta \mathbf{1}_{\{L^j_\vartheta > F^j_\vartheta\}} + L^i_\vartheta \mathbf{1}_{\{L^j_\vartheta = F^j_\vartheta\}}$.

---

[21] An extended mixed strategy can only be superior to a pure one at an opponent's jump $\Delta G^\vartheta_j(\tau)$ that is not terminal, in order to secure the payoff $\Delta G^\vartheta_j(\tau) F^i_\tau + (1 - G^\vartheta_j(\tau)) L^i_\tau$ if this is the unique optimal limit of pure (and therefore of any standard mixed) strategies. That limit is not attainable without extensions if $F^i_\tau > M^i_\tau$.

[22] The proposition can easily be modified to construct equilibria with nobody realizing a first-mover advantage and both players $i = 1, 2$ receiving the payoff F^i_ϑ. The corresponding extensions are $\alpha^\vartheta_i(t) = \mathbf{1}_{\{L^j_t = M^j_t\}} + \mathbf{1}_{\{L^j_t > M^j_t\}} (L^j_t - F^j_t)/(L^j_t - M^j_t)$ if $t = \inf\{u \geq t \,|\, (L^1_u - F^1_u) \wedge (L^2_u - F^2_u) > 0\}$ and $\alpha^\vartheta_i(t) = 0$ else.

Remark 3.2. Note that in particular $\alpha_i^\vartheta(t) = 0$ if $t < \inf\{u \geq t \,|\, (L_u^1 - F_u^1) \wedge (L_u^2 - F_u^2) > 0\}$. On the "boundary" of the preemption region, i.e., if $t = \inf\{u \geq t \,|\, (L_u^1 - F_u^1) \wedge (L_u^2 - F_u^2) > 0\}$ but $(L_t^1 - F_t^1) \wedge (L_t^2 - F_t^2) = 0$, either α_i^ϑ might not be right-continuous for three reasons. First, if we have $L_t^j = F_t^j = M_t^j$, there might not be a right-hand limit $\alpha_i^\vartheta(t+)$, which we can accomodate by setting $\alpha_i^\vartheta(t) = 1$ as player j will be indifferent. Second, in asymmetric models we have to ensure that a player with a strict first-mover advantage $L_t^i - F_t^i > 0$ can realize it by playing $\alpha_i^\vartheta(t) = 1$ and the other playing $\alpha_j^\vartheta(t) = 0$; cf. Theorem 3.3 below. Third, if $L_t^j > F_t^j$ but $L_t^i = F_t^i$, then $\mathbf{1}_{\{L_t^i > F_t^i\}}$ might not have a right-hand limit; in this case we have $\alpha_i^\vartheta(t) = 0$ (and $\alpha_j^\vartheta(t) = 1$).

In the symmetric case, when the payoff processes do not depend on the players' names, each player becomes leader or follower with probability $\frac{1}{2}$ if $L_\vartheta = F_\vartheta > M_\vartheta$, because then the lim inf and lim sup in Definition 2.9 are both $\frac{1}{2}$. This is the same outcome as the result of the Taylor expansion in Fudenberg and Tirole (1985) for their smooth, deterministic model. If $L_\vartheta > F_\vartheta$, there is a positive probability of simultaneous stopping, however, which is the price of mutual preemption.

α_i^ϑ in Proposition 3.1 does not depend on ϑ (except for the requirement $\alpha_i^\vartheta{=}0$ on $[0, \vartheta)$, of course), so applying the construction to any $\vartheta \in \mathscr{T}$ induces a subgame-perfect equilibrium if $L^i > F^i$ almost everywhere for both $i = 1, 2$. Otherwise, if there is not a persistent first-mover advantage, then there can exist many different types of equilibria. One quite general class for which we can use Proposition 3.1 is when the leader's payoff tends to increase in expectation, i.e., when L^i is a submartingale for each player $i = 1, 2$. Then no player wants to stop where $F^i > L^i$, so stopping results only from preemption; see Theorem 3.3.[23]

3.1. *General issues with preemption equilibria*

Preemptive equilibria are central in the strategic real options literature; a simple deterministic example is shown in Figure 1. A number of papers using in fact stochastic models argue that in equilibrium player 1 becomes leader at $\tau^{\mathcal{P}}$, and player 2 becomes follower.[24]

Stopping must occur no later than at $\tau^{\mathcal{P}}$ in equilibrium, because the players would try to preempt each other where both have a strict first-mover advantage, i.e., $L^i > F^i$. In this deterministic example it also seems clear that no player wants to stop at any $t < \tau^{\mathcal{P}}$ because the payoffs keep increasing. There are two general issues in supporting "stopping at $\tau^{\mathcal{P}}$" as an equilibrium.

First, for stopping to occur at $\tau^{\mathcal{P}}$, player 1 must not be able to realize a further increase in L^1. Exploiting the increasing payoff can only be prevented by a (credible) threat of player 2 to stop sufficiently quickly after $\tau^{\mathcal{P}}$ if player 1 does

[23] For the limits of this logic, however, see Section 4.3.
[24] See, e.g., Weeds (2002), Pawlina and Kort (2006), Mason and Weeds (2010).

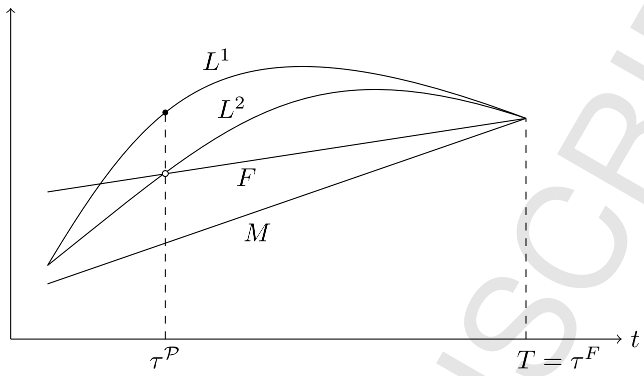

Figure 1: Preemption with asymmetric leader payoffs

not stop. There is no such threat in the mentioned papers.[25] We model a proper game theoretic equilibrium of preemption on $(\tau^{\mathcal{P}}, T)$ with the strategies given in Proposition 3.1, such that the (first mode of the) game ends immediately at any point in this region, even if some player deviates unilaterally. Both players are also willing to stop immediately because the extensions α_i^{ϑ} allow to control the probability of simultaneous stopping, which the players want to avoid.[26] The equilibrium outcome probabilities of who stops first (player 1, player 2, or both) depend on the relation between L^i, F^i and M^i, $i = 1, 2$.

Second, on $[0, \tau^{\mathcal{P}})$ the players must be willing to wait until $\tau^{\mathcal{P}}$. In the deterministic example, no one wants to stop at any $t < \tau^{\mathcal{P}}$ because the payoffs obviously keep increasing. Nevertheless, for waiting *until* $\tau^{\mathcal{P}}$ to be an equilibrium, (i) player 1 must also be sure to become leader and (ii) player 2 must be sure that there is no possibility of simultaneous stopping. We support exactly that outcome in Proposition 3.1.[27] In stochastic models, however, it is much less clear that the corresponding payoffs at $\tau^{\mathcal{P}}$ are in expectation (at least) as good

[25] In particular, in many real option models, the follower still has an investment option and F is the value of investing at some later time τ^F, which is optimal given that the leader already has invested. Thus, any strategy that leads to becoming follower induces the same *outcome* of investing at τ^F (as an implicit reaction when the leader invests). If, however, a player ending up as follower already planned to invest at τ^F from the start, this would not put preemptive pressure on the opponent, so this cannot be an equilibrium *strategy* (for subgames in which nobody has invested, yet).

[26] A simpler alternative approach is to *replace* the simultaneous stopping payoffs if both players try to stop at the same time by some distribution over payoffs that yields at least the follower payoffs in expectation. Grenadier (1996) and Hoppe and Lehmann-Grube (2005), e.g., let each player become leader or follower with probability 1/2 in their symmetric models.

[27] For this outcome we cannot require right-continuity of the $\alpha_i^t(\cdot)$ at $\tau^{\mathcal{P}}$ because then player 2 would become leader and player 1 follower for sure. Indeed, if $t \leq \tau^{\mathcal{P}}$, then $\alpha_1^t(\tau^{\mathcal{P}}+) = \lim_{u \searrow \tau^{\mathcal{P}}} \mathbf{1}_{\{L_u^1 > F_u^1\}} \mathbf{1}_{\{L_u^2 > F_u^2\}} \frac{L_u^2 - F_u}{L_u^2 - M_u} = 0$ and $\alpha_2^t(\tau^{\mathcal{P}}+) = \lim_{u \searrow \tau^{\mathcal{P}}} \mathbf{1}_{\{L_u^1 > F_u^1\}} \mathbf{1}_{\{L_u^2 > F_u^2\}} \frac{L_u^1 - F_u}{L_u^1 - M_u} > 0$.

as becoming leader earlier on. Even if one observes that for typical Markovian models the *current* leader and follower payoffs as functions of the state (instead of time) look similar to Figure 1, discounting and the random time span to reaching $\tau^{\mathcal{P}}$ make intertemporal comparisons much more complex. Therefore one needs to study related optimal stopping problems that are often neglected in the literature.[28] We consider these problems in the following.

3.2. Subgame-perfect preemption equilibria

A reasonable stochastic analog to a deterministically increasing leader payoff is assuming L^i to increase in expectation, i.e., to be a submartingale at least outside the preemption region. Then we get a very general existence result, without any particular assumption on the underlying stochastics. The intuitive result does however rely on continuity; cf. the example in Section 4.3. An alternative argument to obtain such "purely preemptive" equilibria is presented in Section 4.2, based on F^i being the value process of the follower's remaining stopping problem.

Theorem 3.3. *Assume that each L^i is a submartingale, that L^i and F^i are a.s. continuous and that $F^i \geq M^i$, $i = 1, 2$. Then there exists a subgame-perfect equilibrium $\big(G_1, \alpha_1\big)$, $\big(G_2, \alpha_2\big)$ with α_i^ϑ given by Proposition 3.1 and $G_i^\vartheta = \mathbf{1}_{\{t \geq \tau^{\mathcal{P}}(\vartheta)\}}$ for all $\vartheta \in \mathscr{T}$ and $i = 1, 2$.*

The resulting payoffs are

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = \begin{cases} E\big[L^i_{\tau^{\mathcal{P}}(\vartheta)} \,\big|\, \mathscr{F}_\vartheta\big] & \text{if } \{\vartheta < \tau^{\mathcal{P}}(\vartheta)\} \text{ or } L^j_\vartheta = F^j_\vartheta, \\ F^i_\vartheta & \text{else.} \end{cases}$$

Remark 3.4. For equilibria of this type it would suffice that each L^i is a semimartingale, i.e., of the form $L^i = N^i + A^i$ with a martingale N^i and a finite-variation process A^i (e.g., L^i is a diffusion) and that each A^i, which inherits continuity from L^i, is non-decreasing outside the preemption region $\{(L^1 - F^1) \wedge (L^2 - F^2) > 0\}$.

4. Illustrative examples

In the following we present some examples to illustrate the equilibrium concept developed in this paper. Section 4.1 shows the workings of extended mixed strategies in a simple asymmetric game. Section 4.2 presents a standard model from the theory of strategic investment under uncertainty and derives strategies to actually support typical outcomes that are proposed in the literature as (subgame-perfect) equilibria. Finally, Section 4.3 gives an example showing that the logic used in Theorem 3.3 is sensitive to jumps. The focus is here on

[28] A frequent argument to justify waiting is a current second-mover advantage, i.e., $F^i > L^i$, which by itself is not sufficient.

stopping occuring by preemption. For an equilibrium with nondegenerate distributions G_i^ϑ that imply stopping at a rate and that have both times with first- and second-mover advantages in their support see Steg and Thijssen (2015).

4.1. Example: the cross-country grab-the-dollar-game

First, we analyze the stochastic version of the grab-the-dollar game presented as Example 2.4 to illustrate our definition of equilibrium. It shows why it is important to allow for adapted (instead of $\mathscr{F}_\vartheta$-measurable) strategies in the subgames starting at some stopping time ϑ. With stochastic payoffs, it is generally impossible to fix one's own strategy independently of the development of the state variables.

Recall $L_t^1 = \exp(-rt)$ if the American wins the dollar at time $t \geq 0$ and $L_t^2 = X_t \exp(-rt)$ if the European wins (where $X_t > 0$ is the exchange rate), whereas $F_t^1 = F_t^2 = 0$, and the discounted penalty for simultaneous grabbing is $M_t^1 = M_t^2 = -\exp(-rt)$. The following equilibrium is an application of Proposition 3.1 for global first-mover advantages $L^i - F^i > 0$. Feasibility and time-consistency of the strategies are straightforward to verify.

Proposition 4.1. *A subgame-perfect equilibrium for the cross-country grab-the-dollar game is given by the strategies $G_i^\vartheta(t) = \mathbf{1}_{\{t\geq\vartheta\}}$, $i = 1, 2$ ("grab immediately") and*

$$\alpha_1^\vartheta(t) = \frac{X_t}{1 + X_t}\,\mathbf{1}_{\{t\geq\vartheta\}}$$

and

$$\alpha_2^\vartheta(t) = \frac{1}{2}\,\mathbf{1}_{\{t\geq\vartheta\}}$$

for all stopping times $\vartheta \in \mathscr{T}$ and $t \in \mathbb{R}_+$.

This equilibrium is in fact the limit of the discrete-time mixed equilibria discussed in Section 2.1.2, with both players' payoffs being 0, also for any deviation. $\alpha_1^\vartheta(t)$ is increasing in X_t, because in order to keep player 2 indifferent, the probability that he becomes leader must decrease and/or that of a collision must increase, which now both hold. The outcome distribution is that either player 1 wins the dollar at $t = 0$ or a collision happens with probability $X_0/(2X_0 + 1)$ each, and player 2 wins with probability $1/(2X_0 + 1)$.

4.2. Example: preemptive market entry

Now we analyze the market entry game presented as Example 2.5 – which is a typical strategic real option exercise problem – in order to show how to derive (even subgame-perfect) equilibrium strategies for similar models.

In order to obtain possibly explicit results and because this is the most familiar model in the literature, we let the duopoly profit stream after entry by both firms $X = (X_t)_{t\geq 0}$ be a geometric Brownian motion. The idea of proof is very general, however; see Steg (2015a). Hence, assume now that X is the unique strong solution to the stochastic differential equation

$$\frac{dX}{X} = \mu\,dt + \sigma\,dB$$

with given initial value $X_0 = x \in \mathbb{R}_+$, where B is a Brownian motion and μ, σ are some constants. Furthermore, profits are discounted at a common and constant rate $r > \max(\mu, 0)$, which ensures integrability of our payoff processes and finiteness of the subsequent stopping problems.[29] The sunk cost of entry is $I > 0$.

As X is a geometric Brownian motion, the follower's payoff at any $\tau \in \mathscr{T}$ has an explicit representation. Let $\beta_1 > 1$ be the positive root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$ and define

$$x^F = \frac{\beta_1}{\beta_1 - 1}(r-\mu)I > rI.$$

Then it is a standard result from the real options and the optimal stopping literature that the optimal policy for the follower is to invest as soon as the state process X exceeds the threshold x^F, and the associated value is

$$\begin{aligned} F_\tau &:= \sup_{\tau' \geq \tau} E\biggl[\int_{\tau'}^{\infty} e^{-rs}(X_s - rI)\,ds \,\bigg|\, \mathscr{F}_\tau\biggr] \\ &= \begin{cases} e^{-r\tau}\left(\dfrac{X_\tau}{x^F}\right)^{\beta_1}\left(\dfrac{x^F}{r-\mu} - I\right) & \text{if } X_\tau < x^F, \\ e^{-r\tau}\left(\dfrac{X_\tau}{r-\mu} - I\right) & \text{else.} \end{cases} \end{aligned}$$

If the leader invests at any stopping time $\tau \in \mathscr{T}$, we denote the optimal investment time of the follower by

$$\tau^F(\tau) := \inf\{s \geq \tau \mid X_s \geq x^F\}.$$

Then the leader's payoff has a similar explicit representation

$$\begin{aligned} L_\tau &:= E\biggl[\int_{\tau}^{\tau^F(\tau)} e^{-rs}(MX_s - rI)\,ds + \int_{\tau^F(\tau)}^{\infty} e^{-rs}(X_s - rI)\,ds \,\bigg|\, \mathscr{F}_\tau\biggr] \\ &= \begin{cases} e^{-r\tau}\left(\dfrac{MX_\tau}{r-\mu} - I + \left(\dfrac{X_\tau}{x^F}\right)^{\beta_1}\left(\dfrac{x^F(1-M)}{r-\mu}\right)\right) & \text{if } X_\tau < x^F, \\ F_\tau \quad \text{else.} \end{cases} \end{aligned}$$

[29]If $r > \max(\mu, 0)$, then $(e^{-rt}X_t)$ is bounded by an integrable random variable. Indeed, for $\sigma > 0$ we have $\sup_t e^{-rt}X_t = X_0 e^{\sigma Z}$ with $Z = \sup_t B_t - t(r-\mu+\sigma^2/2)/\sigma$, which is exponentially distributed with rate $2(r-\mu)/\sigma + \sigma$ (see, e.g., Revuz and Yor (1999), Exercise (3.12) 4°). Thus, $E[\sup_t e^{-rt}X_t] = X_0(1+\sigma^2/2(r-\mu)) \in \mathbb{R}_+$, implying that $(e^{-rt}X_t)$ is of class (D); analogously for $\sigma < 0$.

Finally, the payoff from simultaneous investment is simply given by

$$\begin{aligned} M_\tau &:= E\left[\int_\tau^\infty e^{-rs}(X_s - rI)\, ds \,\middle|\, \mathscr{F}_\tau\right] \\ &= e^{-r\tau}\left(\frac{X_\tau}{r-\mu} - I\right), \end{aligned}$$

i.e., $M_\tau = F_\tau = L_\tau$ whenever $X_\tau \geq x^F$. Thanks to the explicit representations, we see that it suffices to define the processes (L_t), (F_t) and (M_t) for $t \in \mathbb{R}_+$ (and $L_\infty = F_\infty = M_\infty = 0$) and to evaluate them at any $\tau \in \mathscr{T}$ for obtaining the correct payoffs at stopping times, i.e., which mean that the follower invests at $\tau^F(\tau)$. All processes are (right-)continuous like X.

In order to determine when there is a first- or second-mover advantage, we can rely on the strong Markov property and identify the corresponding regions of the state space of the process X: there exists a unique $x^P \in (0, x^F)$ such that

$$\begin{cases} L_t < F_t & \text{iff } X_t \in [0, x^P), \\ L_t > F_t & \text{iff } X_t \in (x^P, x^F).^{30} \end{cases}$$

Consequently, the interval $\mathcal{P} = (x^P, x^F)$ is the preemption region for the driving process X. On $\{X \in \mathcal{P}\}$ we have equilibria of immediate stopping, with coordination by extended mixed strategies following Proposition 3.1 and with expected payoffs F. Let

$$\tau^{\mathcal{P}}(\vartheta) := \inf\{s \geq \vartheta \mid X_s \in \mathcal{P}\}$$

denote the hitting time of the preemption region after any stopping time $\vartheta \in \mathscr{T}$.

On $\{X_\vartheta < x^P\}$ we have $L_\vartheta < F_\vartheta = E[F_{\tau^{\mathcal{P}}(\vartheta)} | \mathscr{F}_\vartheta] = E[L_{\tau^{\mathcal{P}}(\vartheta)} | \mathscr{F}_\vartheta]$, since F is a martingale in its continuation region up to $\tau^F(\vartheta) > \tau^{\mathcal{P}}(\vartheta)$. Therefore the players are indeed willing to wait until $\tau^{\mathcal{P}}(\vartheta)$, where the equilibrium payoff from preemption is $F_{\tau^{\mathcal{P}}(\vartheta)}$.

Finally, let $\mathcal{M} = [x^F, \infty)$, so that $M_t = F_t$ iff $X_t \in \mathcal{M}$. On $\{X \in \mathcal{M}\}$ we have equilibria of simultaneous stopping. In fact, given that the preemption payoffs on $\{X \in \mathcal{P}\}$ are F, stopping is even strictly dominant on $\{X \in \mathcal{M}\}$, because then the drift of the supermartingale F is the strictly negative forgone

[30] Consider time 0, recalling that $X_0 = x \in \mathbb{R}_+$. We express the dependence on the starting value by writing $L_0 = L(x)$ and $F_0 = F(x)$. Then $L(x) - F(x) \to -I < 0$ for $x \to 0$. Further,

$$\frac{\partial (L(x) - F(x))}{\partial x} = \frac{M}{r-\mu} + \frac{\beta_1}{x^F}\left(\frac{x}{x^F}\right)^{\beta_1 - 1}\left(I - \frac{Mx^F}{r-\mu}\right).$$

The term in the last parentheses is strictly negative by $M > 1$. Thus, the displayed derivative is strictly decreasing in x, starting at $M/(r-\mu) > 0$ for $x = 0$ and ending at $(1-M)(\beta_1 - 1)/(r-\mu) < 0$ for $x = x^F$, where $L(x^F) = F(x^F)$. Hence, there exists a unique $x^P \in (0, x^F)$ such that $L(x) - F(x) < 0$ iff $x \in [0, x^P)$ and $L(x) - F(x) > 0$ iff $x \in (x^P, x^F)$.

revenue $-e^{-rt}(X_t - rI)\,dt$. Thus we obtain the following equilibrium, letting $\tau^{\mathcal{M}}(\vartheta) := \inf\{s \geq \vartheta \mid X_s \in \mathcal{M}\}$ denote the hitting time of $\mathcal{M}$. (See the appendix for details.)

Proposition 4.2. *There exists a subgame-perfect equilibrium for the preemptive market entry game with extended mixed strategies (G_1, α_1) and (G_2, α_2) given as follows. For any $i = 1, 2$ and $\vartheta \in \mathscr{T}$ set*

$$\alpha_i^{\vartheta}(t) = \begin{cases} \dfrac{L_t - F_t}{L_t - M_t} & \text{if} \quad L_t > F_t \Leftrightarrow X_t \in \mathcal{P} = (x^P, x^F), \\ 1 & \text{if} \quad F_t = M_t \Leftrightarrow X_t \in \mathcal{M} = [x^F, \infty), \\ 0 & \text{else} \end{cases}$$

and

$$G_i^{\vartheta}(t) = \mathbf{1}_{\{t \geq \tau^{\mathcal{P}}(\vartheta) \wedge \tau^{\mathcal{M}}(\vartheta)\}}$$

for any $t \in [\vartheta, \infty]$.

The resulting payoffs are $V_i^{\vartheta}(G_i^{\vartheta}, \alpha_i^{\vartheta}, G_j^{\vartheta}, \alpha_j^{\vartheta}) = E\big[F_{\tau^{\mathcal{P}}(\vartheta) \wedge \tau^{\mathcal{M}}(\vartheta)} \,\big|\, \mathscr{F}_{\vartheta}\big]$.

Fully explicitly, we have

$$\alpha_i^{\vartheta}(t) = \frac{\dfrac{MX_t}{r-\mu} - I + \left(\dfrac{X_t}{x^F}\right)^{\beta_1}\left(I - \dfrac{Mx^F}{r-\mu}\right)}{\dfrac{(M-1)X_t}{r-\mu} + \left(\dfrac{X_t}{x^F}\right)^{\beta_1}\left(\dfrac{(1-M)x^F}{r-\mu}\right)}, \qquad X_t \in \mathcal{P}.$$

The extensions $\alpha_i^{\vartheta}(\cdot)$ here inherit continuity from the payoff processes.[31] As remarked more generally in the context of Proposition 3.1, each firm eventually becomes leader or follower with probability $\frac{1}{2}$ in any subgame that starts with $X_{\vartheta} \leq x^P$.

4.3. Sensitivity to jumps

The following simple economic example illustrates the sensitivity of the logic brought forward in Sections 3.1, 3.2 – that stopping is dominated where $L^i < F^i$ and L^i is (strictly) increasing in expectation – to continuity of the payoff processes. We will show that in *any* equilibrium of the example stopping occurs

[31] This is clear except for possibly two cases. If $X_t = x^P$, then $L_t = F_t$ and $\lim_{u \to t} \mathbf{1}_{\{L_u > F_u\}}(L_u - F_u)/(L_u - M_u) = 0$ as $L_t > M_t$ for $X_t \notin \mathcal{M}$. If $X_t = x^F = \partial\mathcal{M}$, then

$$\lim_{u \to t} \alpha_i^{\vartheta}(u) = \lim_{u \to t} \Big[\mathbf{1}_{\{L_u > F_u\}}(L_u - F_u)/(L_u - M_u) + \mathbf{1}_{\{F_u = M_u\}}\Big] = 1 = \alpha_i^{\vartheta}(t),$$

because $\lim_{x \nearrow x^F}(L(x) - F(x))/(L(x) - M(x)) = 1$ by l'Hôpital (with the notation of fn. 30, writing also $M(x) = M_0$ for $X_0 = x$).

strictly before reaching the preemption region (where both players have a first-mover advantage), although the following regularity properties hold: For each $i = 1, 2$ the leader payoff process L^i is a strict submartingale that is further upper-semicontinuous (the usual regularity condition for optimal stopping), and the follower process F^i is continuous; the preemption region is non-empty with probability 1. We will actually construct subgame-perfect equilibria with immediate stopping at any time.

Consider two rival firms that each can make an investment to diversify to a new product. If only one firm invests, it will take up the new market, whereas the other then becomes monopolist for the old product. The latter is worth a net present value c at the time of investment. Initially only firm 1 has developed a profitable technology for switching to the new product, such that it can invest into it at any time. The technology keeps improving, however, so the value of capturing the new market is increasing in time, even if discounted to time 0. Firm 2 initially has an inferior technology, such that investing into the new product would only yield it a net present value 1 at the time of investment. However, firm 2 can catch up to the superior technology at the hazard rate $\lambda > 0$, after which it could realize the same profit as firm 1. As usual, simultaneous investment is the worst outcome.

We model the payoff processes as follows:

$$
\begin{aligned}
L^1 &= \big(a - e^{-rt}b\big)_{t\in[0,\infty)}, && a = 2 + \tfrac{r}{\lambda},\ b = \tfrac{r}{r+\lambda},\\
L^2 &= \big(e^{-rt}\mathbf{1}_{\{t<T\}} + (a - e^{-rt}b)\mathbf{1}_{\{t\geq T\}}\big)_{t\in[0,\infty)}, &&\\
F^1 &= F^2 = \big(e^{-rt}c\big)_{t\in[0,\infty]}, && c \in [1, 1+b),\\
&\geq M^1,\, M^2. &&
\end{aligned}
$$

The choice of the M^i – the payoffs for simultaneous investment – is unimportant as long as there is a (weak) penalty.[32] $r > 0$ is the fixed discount rate. T is the random time at which firm 2 catches up, exponentially distributed with parameter λ, and defined on some stochastic basis $(\Omega, \mathscr{F}, P)$. Assume this is the only uncertainty, i.e., the filtration $\mathbf{F} = (\mathscr{F}_t)_{t\in[0,\infty)}$ is generated by the process $(\mathbf{1}_{\{t\geq T\}})$.

Notice for the sake of the argument that $F^2_t > L^2_t$ on $[0, T)$ for $c > 1$. All processes except for L^2 (and M^1, M^2, if we like) are continuous. L^2 is continuous from the right and upper-semicontinuous from the left, because $a > 2 > (1 + b) \geq e^{-rt}(1 + b)$. L^1 is strictly increasing and also L^2 is a strict submartingale.[33] The key property for the intended result is that L^2 is strictly decreasing up to T and exceeds the expected value of becoming follower at T

[32] Also F^1 can be modified; for the argument we only need $F^1 \leq F^2$.

[33] Fix two times $0 \leq s < t$. On $\{s \geq T\}$ we have $E[L^2_t \,|\, \mathscr{F}_s] - L^2_s = e^{-rs}b - e^{-rt}b > 0$. On $\{s < T\}$,

$$
\begin{aligned}
E[L^2_t \,|\, \mathscr{F}_s] &= e^{-rt}P[T > t \,|\, T > s] + (a - e^{-rt}b)P[T \leq t \,|\, T > s]\\
&= e^{-rt}e^{-\lambda(t-s)} + (a - e^{-rt}b)(1 - e^{-\lambda(t-s)})
\end{aligned}
$$

as we will show, so firm 2 will stop immediately if there is too little chance to realize L^2 after T.

The preemption region is $\{(L^1 - F^1) \wedge (L^2 - F^2) > 0\} = \bigcup_{\omega\in\Omega}[T(\omega), \infty) \subset \Omega \times [0, \infty]$ since $a > b + c \geq e^{-rt}b + e^{-rt}c$. Its hitting time starting from time 0 is

$$\tau^{\mathcal{P}} := \inf\{s \geq 0 \,|\, (L_s^1 - F_s^1) \wedge (L_s^2 - F_s^2) > 0\} = T.$$

Now suppose there is an equilibrium with preemption using extended mixed strategies inside the preemption region, i.e., with continuation payoff F_ϑ^i for each firm $i = 1, 2$ and any stopping time $\vartheta > \tau^{\mathcal{P}} = T$. Assume that at T, however, the firms can agree on events A_1 and A_2 with $P[A_1 \cap A_2] = 0$, such that firm i even obtains L_T^i on A_i by playing $\alpha_i(T) = 1$ and playing $\alpha_i(T) = 0$ on A_j, $i, j \in \{1, 2\}$, $i \neq j$. Let A_i in fact be exactly the event where $\alpha_i(T) = 1$, so i's payoff on A_i^c is F_T^i. If there is no stopping on $[0, T)$, then the expected payoffs are

$$E[L_T^1 \mathbf{1}_{\{A_1\}} + F_T^1 \mathbf{1}_{\{A_1^c\}}] = E[(a - e^{-rT}b)\mathbf{1}_{\{A_1\}} + e^{-rT}c\mathbf{1}_{\{A_1^c\}}]$$

and

$$E[L_T^2 \mathbf{1}_{\{A_2\}} + F_T^2 \mathbf{1}_{\{A_2^c\}}] \leq E[(a - e^{-rT}b)\mathbf{1}_{\{A_1^c\}} + e^{-rT}c\mathbf{1}_{\{A_1\}}].$$

The estimate follows from $L_T^2 > F_T^2$ and $P[A_2 \setminus A_1^c] = P[A_1 \setminus A_2^c] = 0$. The sum of the expected payoffs is not more than $a - bE[e^{-rT}] + cE[e^{-rT}] = a - (b - c)\lambda/(r + \lambda) < a - b + 1 = L_0^1 + L_0^2$, contradicting the hypothesized equilibrium.

Now suppose there is an equilibrium with mixed strategies G_i^T in the preemption region. From T onwards, the payoff processes are deterministic and continuous. Hendricks and Wilson (1992) show that mixed equilibrium strategies must satisfy

$$\frac{dG_i^T}{1 - G_i^T} = \frac{dL^j}{L^j - F^j} = \frac{re^{-rt}b\,dt}{a - e^{-rt}(b + c)}, \qquad i, j \in \{1, 2\},\, i \neq j,$$

where continuous mixing occurs. We have $0 < (a - e^{-rt}(b + c))^{-1} \leq (a - b - c)^{-1} < \lambda(r + \lambda)/(r^2 + \lambda^2) < \infty$ and thus $\int_T^\infty (1 - G_i^T)^{-1}\, dG_i^T < \infty$, $i = 1, 2$, so there would remain some mass for stopping at $t = \infty$ for both firms. However, $M_\infty^i \leq F_\infty^i = 0 < \lim_{t\nearrow\infty} L_t^i = a$ for both $i = 1, 2$, so there cannot be joint stopping at $t = \infty$.

It follows from Theorems 2 and 3 in Hendricks and Wilson (1992) that at T there only exist equilibria with at least one firm i stopping immediately, i.e.,

---

and thus

$$\begin{aligned}
\partial E[L_t^2 \,|\, \mathscr{F}_s]/\partial t = -(r + \lambda)e^{-rt}e^{-\lambda(t-s)}(1 + b) + a\lambda e^{-\lambda(t-s)} + re^{-rt}b > 0 \quad \forall t > s \geq 0 \\
\Leftrightarrow \quad a\lambda e^{rt} + re^{\lambda(t-s)}b > (r + \lambda)(1 + b) \quad \forall t > s \geq 0 \\
\Leftrightarrow \quad a\lambda + rb \geq (r + \lambda)(1 + b) \\
\Leftrightarrow \quad a \geq (1 + b) + \tfrac{r}{\lambda}.
\end{aligned}$$

Hence, $E[L_t^2 \,|\, \mathscr{F}_s] > \lim_{u\searrow s} E[L_u^2 \,|\, \mathscr{F}_s] = e^{-rs} = L_s^2$.

$\max\{G_1^T(T), G_2^T(T)\} = 1$. A firm that stops receives not more than L_T^i and the respective other receives F_T^j (possibly from simultaneous stopping if $F_T^i = M_T^i$, $i = 1, 2$). Now the previous argument based on the sets $A_i = \{G_i^T(T) = 1\}$ applies again, so stopping must occur strictly before T in equilibrium.

The estimate above extends to any time $t \in [0, T)$, taking conditional expectations. In fact, there exist the following subgame-perfect equilibria with either firm stopping immediately on $[0, T)$. Set

$$\alpha_1^\vartheta(t) = \mathbf{1}_{\{t \geq T\}} \frac{L_t^2 - F_t^2}{L_t^2 - M_t^2} \quad \text{and} \quad \alpha_2^\vartheta(t) = \mathbf{1}_{\{t \geq T\}} \frac{L_t^1 - F_t^1}{L_t^1 - M_t^1}$$

for all stopping times ϑ and $t \in [0, \infty)$. Now pick $i, j \in \{1, 2\}$, $i \neq j$ and set $G_i^\vartheta = \mathbf{1}_{\{t \geq \vartheta\}}$ (stop immediately) and $G_j^\vartheta = \mathbf{1}_{\{t \geq \vartheta \vee T\}}$ (stop at preemption region) for all ϑ. On $\{\vartheta \geq T\}$ there is preemption with payoffs F_ϑ^1, F_ϑ^2. On $\{\vartheta < T\}$, j cannot deviate profitably because $M^j \leq F^j$. Firm i could wait until any $\tau \geq \vartheta$ to obtain $L_\tau^i \mathbf{1}_{\{\tau < T\}} + F_T^i \mathbf{1}_{\{\tau \geq T\}}$. The process $(L_t^i \mathbf{1}_{\{t < T\}} + F_T^i \mathbf{1}_{\{t \geq T\}})_{t \geq 0}$ is however a strict supermartingale for both $i = 1, 2$ with our parametrization.[34] Thus, stopping immediately is indeed optimal.

5. Outlook

Having illustrated in Section 4 how our framework can be used for a rigorous analysis of typical preemption models, we now point out how new results can be obtained that extend the existing literature on timing games. This will also concern games with more than two players.

The examples considered here have involved stopping only due to preemption (except for the last model with Poisson-type uncertainty), and whenever stopping has occurred with some probability, it has with probability one. This is a typical observation for many applications, but of course a very special case.

In general, if Proposition 3.1 is applied to model preemption where both players have a first-mover advantage, one has to determine if any player wants to stop before such a point is reached, which amounts to solving *contrained* optimal stopping problems. Different approaches are available. If the timing game is symmetric, the characterization of solutions from the general theory of optimal stopping (in terms of the *Snell envelope* as in El Karoui (1981), e.g.) can be used to establish the existence and even a meaningful characterization of subgame-perfect equilibria under extremely general assumptions on the payoff processes, see Steg (2015b). Previously, there have not been any general existence results

[34] Fix two times $0 \leq s < t$. On $\{s < T\}$ we have $E[L_t^1 \mathbf{1}_{\{t<T\}} + F_T^1 \mathbf{1}_{\{t \geq T\}} \,|\, \mathscr{F}_s] = (a - e^{-rt}b)e^{-\lambda(t-s)}$ (for $F^1 \equiv 0$) and

$$E[L_t^2 \mathbf{1}_{\{t<T\}} + F_T^2 \mathbf{1}_{\{t \geq T\}} \,|\, \mathscr{F}_s] = e^{-rt} e^{-\lambda(t-s)} + e^{-rs} c \tfrac{\lambda}{r+\lambda} (1 - e^{-(r+\lambda)(t-s)})$$

(resp. $E[L_t^1 \mathbf{1}_{\{t<T\}} + F_T^1 \mathbf{1}_{\{t \geq T\}} \,|\, \mathscr{F}_s] \leq (a - e^{-rt}b)e^{-\lambda(t-s)} + e^{-rs} c \tfrac{\lambda}{r+\lambda} (1 - e^{-(r+\lambda)(t-s)})$ for any other choice of $F^1 \leq F^2$). Now one can continue as in fn. 33.

for subgame-perfect equilibria of stochastic timing games. The G_i^ϑ play a crucial role for symmetric equilibria and will be nondegenerate in many cases.

If the uncertainty results from a Markov process, one can use PDE methods to analyze the stopping problems and, thus, equilibria. Steg and Thijssen (2015) propose a real option model with a two-dimensional state process which randomly generates first- and second-mover advantages. Their equilibrium characterization involves the solution of a free boundary problem, and the players stop with a Markovian hazard rate in a certain region of the state space.

Finally, we now demonstrate that the concept of strategies and equilibrium we have chosen can produce results where other generalizations of Nash equilibrium are known to fail. Specifically, we construct a subgame-perfect equilibrium for a three-player timing game from Laraki et al. (2005), who show that this game has no ε-equilibrium for $\varepsilon > 0$ small enough (whereas any deterministic two-player timing game has a subgame-perfect ε-equilibrium).

The payoffs do not depend on when the first player stops, but only on who stops first, possibly simultaneously. To describe the coordination conflict, imagine the players sitting on a circle, player 3 to the right of player 2 to the right of player 1. Now player $i+1$ or $i+2$ is going to mean the player one or two positions to the right of player i. If one player i stops before $i+1$ and $i+2$, the respective payoffs are $u^i_{\{i\}} = 1$, $u^{i+1}_{\{i\}} = 0$ and $u^{i+2}_{\{i\}} = -1$. If two players i and $i+1$ stop simultaneously before player $i+2$, the respective payoffs are $u^i_{\{i,i+1\}} = 0$, $u^{i+1}_{\{i,i+1\}} = -1$ and $u^{i+2}_{\{i,i+1\}} = 1$. If all players stop simultaneously, each receives 0. These payoffs are zero-sum and imply that whatever group happens to stop first, some player will want to deviate by quitting or joining.

Laraki et al. (2005) argue that even ε-equilibria in mixed strategies fail to exist. We now apply extended mixed strategies and argue that a subgame-perfect equilibrium exists with each player i using $\alpha_i^\vartheta(t) = \frac{1}{2} > 0$ for all $t \geq \vartheta$. To do so we refrain from extending Definition 2.9 to more than two players for all eventualities under our weak regularity conditions. The reason is that given the strictly positive extensions and that only unilateral deviations need to be considered, we can rely on the definitions for two players and Proposition 3.1.[35]

Specifically, we focus on whether player i, when deviating, becomes single

[35]It is straightforward to extend Definition 2.9 if, as in the current case, some player's extension starts with a strictly positive value, and if one demands the extensions to have right-hand limits. Then the present equilibrium can be easily verified directly. For instance, the outcome probability that the group of players who become (joint) leaders is a given set $I \subseteq \{1,2,3\}$ is again that from an infinitely repeated game with constant stage stopping probabilities $\alpha_i^\vartheta(\vartheta)$. Writing $\alpha_i^\vartheta(\vartheta) = s_i$ for brevity, the probability is given by

$$\prod_{i\in I} s_i \prod_{j\in\{1,2,3\}\setminus I}(1-s_j)\cdot\left(\sum_{n=0}^{\infty}\prod_{j=1}^{3}(1-s_j)^n\right) = \prod_{i\in I} s_i \prod_{j\in\{1,2,3\}\setminus I}(1-s_j)\cdot\left(1-\prod_{j=1}^{3}(1-s_j)\right)^{-1}.$$

This formula also has to be amended by a "first round" if some player uses either an isolated mass point $\Delta G_i^\vartheta(\vartheta) \in (0,1)$ or $\alpha_i^\vartheta(\vartheta) = 0 < \alpha_i^\vartheta(\vartheta+)$, in which case one sets $s_i = \alpha_i^\vartheta(\vartheta+)$ in the formula.

leader, joint leader or follower, because given our (limit) interpretation of the extensions, the probabilities for whom of players $i+1$ and $i+2$ belongs to the leaders conditional on at least one of them doing so are given by (abusing notation for brevity)

$$\big[1-\big(1-\alpha^{\vartheta}_{i+1}\big)\big(1-\alpha^{\vartheta}_{i+2}\big)\big]^{-1}\cdot\Big(\alpha^{\vartheta}_{i+1}\big(1-\alpha^{\vartheta}_{i+2}\big),\ \big(1-\alpha^{\vartheta}_{i+1}\big)\alpha^{\vartheta}_{i+2},\ \alpha^{\vartheta}_{i+1}\alpha^{\vartheta}_{i+2}\Big),$$

i.e., presently $\frac{1}{3}$ for each scenario. Thus, player i's expected payoffs conditional on becoming single leader, joint leader, or follower are 1, $\frac{1}{3}(0-1+0)=-\frac{1}{3}$ and $\frac{1}{3}(0-1+1)=0$, respectively, and facing two opponents with strictly positive extensions is the same as facing one opponent with extension values $1-\big(1-\alpha^{\vartheta}_{i+1}\big)\big(1-\alpha^{\vartheta}_{i+2}\big)=\alpha^{\vartheta}_{i+1}+\alpha^{\vartheta}_{i+2}-\alpha^{\vartheta}_{i+1}\alpha^{\vartheta}_{i+2}=\frac{3}{4}$ and payoffs $L^i_\vartheta=1$, $M^i_\vartheta=-\frac{1}{3}$, and $F^i_\vartheta=0$. As $\frac{3}{4}=\frac{1-0}{1-(-1/3)}$, Proposition 3.1 implies that $\alpha^{\vartheta}_i(t)=\frac{1}{2}\mathbf{1}_{\{t\geq\vartheta\}}$ (and hence $G^{\vartheta}_i(t)=\mathbf{1}_{\{t\geq\vartheta\}}$) is optimal for player i. The resulting expected payoff is zero. Time consistency holds trivially with constant positive extension, so we obtain a subgame-perfect equilibrium.

Appendix A. Proofs

Proof of Proposition 3.1. By construction, G^{ϑ}_i and α^{ϑ}_i are a.s. $[0,1]$-valued. G^{ϑ}_i is right-continuous, non-decreasing, attaining 1 where $\alpha^{\vartheta}_i(t)>0$, $t\geq\vartheta$. α^{ϑ}_i takes values in $(0,1)$ only where $(L^1_t-F^1_t)\wedge(L^2_t-F^2_t)>0$, where it is indeed right-continuous.

G^{ϑ}_i is adapted. α^{ϑ}_i is progressively measurable because we can represent it using the process $(\mathbf{1}_{\{t=\inf\{u\geq t\,|\,(L^1_u-F^1_u)\wedge(L^2_u-F^2_u)>0\}\}})_{t\geq0}$, which is the upper-right-continuous modification of the optional process $(\mathbf{1}_{\{(L^1_t-F^1_t)\wedge(L^2_t-F^2_t)>0\}})_{t\geq0}$ and therefore progressively measurable by Theorem IV.33 (c) in Dellacherie and Meyer (1978) as our filtration $\mathbf{F}$ satisfies the usual conditions.

By $\vartheta=\tau^{\mathcal{P}}(\vartheta)$ we also have $\vartheta=\hat{\tau}^{\vartheta}_i=\inf\{u\geq\vartheta\,|\,\alpha^{\vartheta}_i(u)>0\}$ and $L^i_\vartheta\geq F^i_\vartheta$ a.s., $i=1,2$. By our observations about (3.2) we have indifference if $(L^1_\vartheta-F^1_\vartheta)\wedge(L^2_\vartheta-F^2_\vartheta)>0$, implying payoff F^1_ϑ, resp. F^2_ϑ, and it only remains to verify that each player i obtains $\max\{F^i_\vartheta,\alpha^{\vartheta}_j(\vartheta)M^i_\vartheta+(1-\alpha^{\vartheta}_j(\vartheta))L^i_\vartheta\}$ in the cases $L^1_\vartheta=F^1_\vartheta$ or $L^2_\vartheta=F^2_\vartheta$. Now fix $i,j\in\{1,2\}$, $i\neq j$. Consider first $L^i_\vartheta=F^i_\vartheta$. If $L^j_\vartheta>F^j_\vartheta$, then $\alpha^{\vartheta}_j(\vartheta)=1$, so $\alpha^{\vartheta}_i(\vartheta)=0$ is optimal. If also $L^j_\vartheta=F^j_\vartheta$, then still $\alpha^{\vartheta}_j(\vartheta)=1$ if $F^i_\vartheta=M^i_\vartheta$ and player i is indifferent. If $L^j_\vartheta=F^j_\vartheta$ and $F^i_\vartheta>M^i_\vartheta$, then $\alpha^{\vartheta}_j(\vartheta)=\alpha^{\vartheta}_j(\vartheta+)=0$. Now $\alpha^{\vartheta}_i(\vartheta)=1$ is optimal if $F^j_\vartheta=M^j_\vartheta$, and $\alpha^{\vartheta}_i(\vartheta)=\alpha^{\vartheta}_i(\vartheta+)=0$ is optimal if $F^j_\vartheta>M^j_\vartheta$ as then $\lambda^{\vartheta}_M=0$. In all these cases the payoff is $F^i_\vartheta=L^i_\vartheta$.

Finally consider $L^i_\vartheta>F^i_\vartheta$ and $L^j_\vartheta=F^j_\vartheta$. Then $\alpha^{\vartheta}_j(\vartheta)=0$ and thus $\alpha^{\vartheta}_i(\vartheta)=1$ is optimal. In this case the payoff is L^i_ϑ. □

Proof of Theorem 3.3. Admissibility of the strategies for any $\vartheta\in\mathscr{T}$ is obtained as in the proof of Proposition 3.1 and time-consistency is obvious. For any $\vartheta\in\mathscr{T}$ and $i=1,2$, $\big(G^{\vartheta}_i,\alpha^{\vartheta}_i\big)$ is also admissible for the subgame starting at $\tau^{\mathcal{P}}(\vartheta)$ and Proposition 3.1 shows that $\big(G_1^\vartheta, \alpha_1^\vartheta\big)$ and $\big(G_2^\vartheta, \alpha_2^\vartheta\big)$ are mutual best replies at $\tau^{\mathcal{P}}(\vartheta)$. For $\{\vartheta = \tau^{\mathcal{P}}(\vartheta)\}$ this directly implies optimality. For $\{\vartheta < \tau^{\mathcal{P}}(\vartheta)\}$ and any admissible $\big(G_a^\vartheta, \alpha_a^\vartheta\big)$, time consistency and iterated expectations yield the estimate

$$\begin{aligned}
&V_i^\vartheta\big(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\\
&\leq V_i^\vartheta\big(G_a^\vartheta \mathbf{1}_{\{t<\tau^{\mathcal{P}}(\vartheta)\}} + \underbrace{\mathbf{1}_{\{t\geq\tau^{\mathcal{P}}(\vartheta)\}}}_{= \mathbf{1}_{\{t\geq\tau^{\mathcal{P}}(\vartheta)\}}\left[G_a^\vartheta(\tau^{\mathcal{P}}(\vartheta)-) + (1-G_a^\vartheta(\tau^{\mathcal{P}}(\vartheta)-))G_i^\vartheta\right]}, \alpha_a^\vartheta \mathbf{1}_{\{t<\tau^{\mathcal{P}}(\vartheta)\}} + \alpha_i^\vartheta \mathbf{1}_{\{t\geq\tau^{\mathcal{P}}(\vartheta)\}}, G_j^\vartheta, \alpha_j^\vartheta\big),
\end{aligned}$$

$i, j \in \{1,2\}$, $i \neq j$. Furthermore, since $\mathbf{1}_{\{t<\tau^{\mathcal{P}}(\vartheta)\}}\Delta G_j^\vartheta(t) \equiv 0$,

$$V_i^\vartheta\big(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big) = V_i^\vartheta\big(G_a^\vartheta, \alpha_a^\vartheta \mathbf{1}_{\{t\geq\tau^{\mathcal{P}}(\vartheta)\}}, G_j^\vartheta, \alpha_j^\vartheta\big).$$

These two facts, together with $G_j^\vartheta(\tau^{\mathcal{P}}(\vartheta)-) = 0$ and a change of variable as in Lemma B.2, yield the estimate

$$\begin{aligned}
&V_i^\vartheta\big(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\\
&\leq V_i^\vartheta\big(G_a^\vartheta \mathbf{1}_{\{t<\tau^{\mathcal{P}}(\vartheta)\}} + \mathbf{1}_{\{t\geq\tau^{\mathcal{P}}(\vartheta)\}}, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\\
&= E\bigg[\int_{[0,\tau^{\mathcal{P}}(\vartheta))} L_s^i \, dG_a^\vartheta(s) + (1-G_a^\vartheta(\tau^{\mathcal{P}}(\vartheta)-))V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\,\bigg|\,\mathscr{F}_\vartheta\bigg]\\
&= \int_0^1 E\Big[L^i_{\tau_a^G(x)} \mathbf{1}_{\{\tau_a^G(x)\in[0,\tau^{\mathcal{P}}(\vartheta))\}}\,\Big|\,\mathscr{F}_\vartheta\Big]\,dx\\
&+ \int_0^1 E\Big[V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\mathbf{1}_{\{\tau_a^G(x)\in[\tau^{\mathcal{P}}(\vartheta),\infty]\}}\,\Big|\,\mathscr{F}_\vartheta\Big]\,dx\\
&\leq \operatorname*{ess\,sup}_{\tau\geq\vartheta} E\Big[L_\tau^i \mathbf{1}_{\{\tau<\tau^{\mathcal{P}}(\vartheta)\}} + V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\mathbf{1}_{\{\tau\geq\tau^{\mathcal{P}}(\vartheta)\}}\,\Big|\,\mathscr{F}_\vartheta\Big].
\end{aligned}$$

Fubini's Theorem is applied in the second to last step. At $\tau^{\mathcal{P}}(\vartheta)$, the optimal payoff is

$$V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big) = F^i_{\tau^{\mathcal{P}}(\vartheta)} \mathbf{1}_{\{L^j_{\tau^{\mathcal{P}}(\vartheta)} > F^j_{\tau^{\mathcal{P}}(\vartheta)}\}} + L^i_{\tau^{\mathcal{P}}(\vartheta)} \mathbf{1}_{\{L^j_{\tau^{\mathcal{P}}(\vartheta)} = F^j_{\tau^{\mathcal{P}}(\vartheta)}\}}.$$

If L^i and F^i are continuous, $i = 1,2$, then $(L^1_{\tau^{\mathcal{P}}(\vartheta)} - F^1_{\tau^{\mathcal{P}}(\vartheta)}) \wedge (L^2_{\tau^{\mathcal{P}}(\vartheta)} - F^2_{\tau^{\mathcal{P}}(\vartheta)}) = 0$ on $\{\vartheta < \tau^{\mathcal{P}}(\vartheta)\}$, a.s. Hence, for the given strategies,

$$\begin{aligned}
&V_i^\vartheta\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\\
&= E\Big[V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\,\Big|\,\mathscr{F}_\vartheta\Big]\\
&= E\Big[L^i_{\tau^{\mathcal{P}}(\vartheta)} \mathbf{1}_{\{L^j_{\tau^{\mathcal{P}}(\vartheta)} > F^j_{\tau^{\mathcal{P}}(\vartheta)}\}} + L^i_{\tau^{\mathcal{P}}(\vartheta)} \mathbf{1}_{\{L^j_{\tau^{\mathcal{P}}(\vartheta)} = F^j_{\tau^{\mathcal{P}}(\vartheta)}\}}\,\Big|\,\mathscr{F}_\vartheta\Big]\\
&= E\Big[L^i_{\tau^{\mathcal{P}}(\vartheta)}\,\Big|\,\mathscr{F}_\vartheta\Big] = \operatorname*{ess\,sup}_{\tau\geq\vartheta} E\Big[L^i_{\tau\wedge\tau^{\mathcal{P}}(\vartheta)}\,\Big|\,\mathscr{F}_\vartheta\Big]\\
&= \operatorname*{ess\,sup}_{\tau\geq\vartheta} E\Big[L_\tau^i \mathbf{1}_{\{\tau<\tau^{\mathcal{P}}(\vartheta)\}} + V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\mathbf{1}_{\{\tau\geq\tau^{\mathcal{P}}(\vartheta)\}}\,\Big|\,\mathscr{F}_\vartheta\Big].
\end{aligned}$$

Therefore, player i has no incentive to change the strategy $\big(G_i^\vartheta, \alpha_i^\vartheta\big)$ by placing any mass on $[\vartheta, \tau^{\mathcal{P}}(\vartheta))$. $\square$

Proof of Proposition 4.2. The proof is analogous to that of Theorem 3.3, except that we use the martingale property of F in its continuation region $\{X \in [0, x^F)\}$ instead of a submartingale property of L.[36] Concerning the payoffs, fix $\vartheta \in \mathscr{T}$. If $X_\vartheta \geq x^P$, then the payoffs to both players are $V_i^\vartheta\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big) = F_\vartheta$ and optimality follows again from Proposition 3.1. If $X_\vartheta < x^P$, i.e., $\vartheta < \tau^{\mathcal{P}}(\vartheta)$, then we replace the estimate in the proof of Theorem 3.3 by

$$
\begin{aligned}
&V_i^\vartheta\big(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\\
&\leq V_i^\vartheta\big(G_a^\vartheta \mathbf{1}_{\{t<\tau^{\mathcal{P}}(\vartheta)\}} + \mathbf{1}_{\{t \geq \tau^{\mathcal{P}}(\vartheta)\}}, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\\
&= E\bigg[\int_{[0,\tau^{\mathcal{P}}(\vartheta))} L_s\, dG_i^\vartheta(s) + (1 - G_a^\vartheta(\tau^{\mathcal{P}}(\vartheta)-))V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\,\bigg|\,\mathscr{F}_\vartheta\bigg]\\
&= \int_0^1 E\bigg[L_{\tau_a^G(x)} \mathbf{1}_{\{\tau_a^G(x) \in [0,\tau^{\mathcal{P}}(\vartheta))\}}\,\bigg|\,\mathscr{F}_\vartheta\bigg]\, dx\\
&+ \int_0^1 E\bigg[V_i^{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\mathbf{1}_{\{\tau_a^G(x) \in [\tau^{\mathcal{P}}(\vartheta),\infty]\}}\,\bigg|\,\mathscr{F}_\vartheta\bigg]\, dx\\
&\leq \int_0^1 E\bigg[F_{\tau_a^G(x)} \mathbf{1}_{\{\tau_a^G(x) \in [0,\tau^{\mathcal{P}}(\vartheta))\}}\,\bigg|\,\mathscr{F}_\vartheta\bigg]\, dx\\
&+ \int_0^1 E\bigg[F_{\tau^{\mathcal{P}}(\vartheta)}\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big)\mathbf{1}_{\{\tau_a^G(x) \in [\tau^{\mathcal{P}}(\vartheta),\infty]\}}\,\bigg|\,\mathscr{F}_\vartheta\bigg]\, dx\\
&= E\big[F_{\tau^{\mathcal{P}}(\vartheta)}\,\big|\,\mathscr{F}_\vartheta\big] = V_i^\vartheta\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big).
\end{aligned}
$$

$\square$

Appendix B. Technical results

Lemma B.1. *A measurable process $X = (X_t)_{t\in\mathbb{R}_+}$ is of class* (D) *iff the set $\{X_\tau \,|\, \tau$ a stopping time$\}$ is uniformly integrable for any given $X_\infty \in L^1(P)$.*

Proof. We only need to show necessity. Let X be of class (D), fix arbitrary $X_\infty \in L^1(P)$, and let $\mathscr{T}$ denote the set of all stopping times. Then, for any $\tau \in \mathscr{T}$ and $n \in \mathbb{N}$, $|X_{\tau\wedge n}|\,\mathbf{1}_{\{\tau<\infty\}} \leq |X_{\tau\wedge n}|$. The set $\{|X_{\tau\wedge n}|\,\mathbf{1}_{\{\tau<\infty\}} \,|\, \tau \in \mathscr{T}, n \in \mathbb{N}\} \cup \{|X_\tau| \,|\, \tau \in \mathscr{T}, \tau < \infty\}$ is thus uniformly integrable, too. As we may also include limits in probability of its elements, and $|X_\tau|\,\mathbf{1}_{\{\tau<\infty\}} = \lim_{n\to\infty} |X_{\tau\wedge n}|\,\mathbf{1}_{\{\tau<\infty\}}$ a.s. for any $\tau \in \mathscr{T}$, we observe that $\{|X_\tau|\,\mathbf{1}_{\{\tau<\infty\}} \,|\, \tau \in \mathscr{T}\}$ is uniformly integrable.

For $X_\infty \in L^1(P)$, also $\{|X_\infty|\,\mathbf{1}_{\{\tau=\infty\}} \,|\, \tau \in \mathscr{T}\}$ is uniformly integrable. Now let $\varepsilon > 0$. By uniform integrability there exists $\delta > 0$ such that for any measurable A with $P(A) < \delta$ and any $\tau \in \mathscr{T}$ it holds that $\max\big\{E[|X_\tau|\,\mathbf{1}_{\{\tau<\infty\}}\mathbf{1}_{\{A\}}], E[|X_\infty|\,\mathbf{1}_{\{\tau=\infty\}}\mathbf{1}_{\{A\}}]\big\} \leq$

[36] L need not be a submartingale outside the preemption region, depending on the parameter values. In particular one can show that if $\mu \leq 0$, the drift of L is strictly negative for all M sufficiently close to 1 and X_t sufficiently close to x^P.

$\frac{\varepsilon}{2}$. Therefore $E[(|X_\tau|\,\mathbf{1}_{\{\tau<\infty\}}+|X_\infty|\,\mathbf{1}_{\{\tau=\infty\}})\mathbf{1}_{\{A\}}]\le\varepsilon$, showing that $\{|X_\tau|\,\mathbf{1}_{\{\tau<\infty\}}+|X_\infty|\,\mathbf{1}_{\{\tau=\infty\}}\,|\,\tau\in\mathscr{T}\}$ is uniformly integrable as claimed. □

Lemma B.2. *If L is a (measurable) process of class* (D) *then there exists a constant $K\in\mathbb{R}_+$ such that for any process G that is a.s. right-continuous, non-decreasing, non-negative and bounded by some $G_\infty\in L^\infty(P)$ and all random variables $0\le a\le b\le\infty$ a.s. it holds that*

(1)
$$E\Big[\int_{[a,b)}|L_t|\;dG_t\Big]\le K\|G_\infty\|_\infty<\infty$$

and

(2)
$$\int_{[a,b)}|L_t|\;dG_t=\int_0^\infty\big|L_{\tau^G(x)}\big|\mathbf{1}_{\{\tau^G(x)\in[a,b)\}}\,dx<\infty\quad a.s.,$$

where $\tau^G(x):=\inf\{t\ge0\,|\,G_t\ge x\}$, $x\in\mathbb{R}_+$, and $\Delta G_0\equiv G_0$; equivalently, "$G_t>x$" in $\tau^G(x)$.

If $\{|L_\tau|\,\mathbf{1}_{\{\tau<\infty\}}\,|\,\tau\in\mathscr{T}\}$ is bounded in $L^\infty(P)$ by $K\in\mathbb{R}_+$ and G bounded by some $G_\infty\in L^1(P)$, then (1) *holds with $KE[G_\infty]$ instead and* (2) *as stated.*

Proof. The (a.s.) non-decreasing family of stopping times $\big(\tau^G(x)\big)_{x\in\mathbb{R}_+}$ is the left-continuous inverse of G, which satisfies

$$\tau^G(x)\le t\;\Leftrightarrow\;G_t\ge x.$$

Thus, with the convention $\int_{[0,c]}dG=G_c$, $\int_{[0,\infty)}\mathbf{1}_{\{A\}}\,dG=\int_0^\infty\mathbf{1}_{\{\tau^G(x)\in A\}}\,dx$ for all $A\in\{[0,c]\,|\,c\in\mathbb{R}_+\}$ and also for $A=\mathbb{R}_+$ (by monotone convergence) a.s. By a monotone class argument[37] the relation holds for all Borel sets from $\mathbb{R}_+$ a.s.

Since $L_\cdot(\omega)\colon\mathbb{R}_+\to\mathbb{R}$, $t\mapsto L_t(\omega)$, is Borel measurable[38] like the function $\mathbf{1}_{\{t\in[a(\omega),b(\omega))\}}$, we now obtain the following change-of-variable formula[39]:

$$\int_{[a,b)}|L_t|\;dG_t=\int_{\{\tau^G(x)<\infty\}}\big|L_{\tau^G(x)}\big|\mathbf{1}_{\{\tau^G(x)\in[a,b)\}}\,dx\qquad\text{a.s.}$$

As $\inf\{t\ge0\,|\,G_t>x\}=\tau^G(x+)$, which differs from $\tau^G(x)$ only on a set of Lebesgue measure zero, we can equivalently use the former. By Fubini's Theorem

$$E\Big[\int_0^\infty\big|L_{\tau^G(x)}\big|\mathbf{1}_{\{\tau^G(x)\in[a,b)\}}\,dx\Big]\le\int_0^{\|G_\infty\|_\infty}E\Big[\big|L_{\tau^G(x)}\big|\mathbf{1}_{\{\tau^G(x)<\infty\}}\Big]\,dx.\tag{B.1}$$

[37]See, e.g., Kallenberg (2002), Theorem 1.1.
[38]See, e.g., Kallenberg (2002), Lemma 1.26 (i).
[39]See, e.g., Kallenberg (2002), Lemma 1.22. One needs to restrict dx to $\{\tau^G(x)<\infty\}$, which is redundant when integrating over $[a,b)$.

As L is of class (D), $\{|L_\tau|\,\mathbf{1}_{\{\tau<\infty\}}\,|\,\tau\in\mathscr{T}\}$ is bounded in $L^1(P)$ by some $K<\infty$, whence the RHS of (B.1) is bounded by $K\|G_\infty\|_\infty$ if the latter is finite. If $\sup_{\tau\in\mathscr{T}}\|L_\tau\mathbf{1}_{\{\tau<\infty\}}\|_\infty\le K$ and G bounded by $G_\infty\in L^1(P)$, then the RHS of (B.1) is bounded by

$$\int_0^\infty E\Big[K\mathbf{1}_{\{\tau^G(x)<\infty\}}\Big]\,dx=E\bigg[\int_0^\infty K\mathbf{1}_{\{\tau^G(x)<\infty\}}\,dx\bigg]\le KE[G_\infty]<\infty.$$

In either case it follows that $\int_{[a,b)}|L_t|\,dG_t<\infty$ a.s. □

Lemma B.3. *Suppose α is a progressively measurable process and ${}^o\alpha$ its optional projection.*[40] *Let $\tau\in\mathscr{T}$ be given. Then ${}^o\alpha$ is a.s. right-continuous at $\tau<\infty$ where α is so.*

Proof. For any $\varepsilon>0$ define $\tau_\varepsilon:=\inf\{t\ge\tau\,|\,|\alpha_t-\alpha_\tau|>\varepsilon\}\in\mathscr{T}$ as α is progressive. Then the set $B_\varepsilon:=\{(\omega,t)\,|\,\tau\le t<\infty,{}^o\alpha_t-{}^o\alpha_\tau>\varepsilon\}\cap[\tau,\tau_\varepsilon)$ is optional and $P[\sigma\in B_\varepsilon]=0$ for any $\sigma\in\mathscr{T}$ because $\alpha_\sigma={}^o\alpha_\sigma$ on $\{\sigma<\infty\}$ a.s. Hence, if we denote by A_ε the canonical projection of B_ε onto Ω, $P[A_\varepsilon]=0$ by the optional section theorem (Dellacherie and Meyer (1978), Theorem IV.84), and therefore $A^c:=\bigcap_{n\in\mathbb{N}}A^c_{1/n}$ is an a.s. event with $(A^c\times\mathbb{R}_+)\cap B_{1/n}=\emptyset$, $n\in\mathbb{N}$. By switching signs we obtain the same result for $|{}^o\alpha_t-{}^o\alpha_\tau|>1/n$ in $B_{1/n}$ (we do not rename any sets).

Now, given any $\omega\in A^c$ for which α is right-continuous at τ, we must have $\tau_{1/\lceil\varepsilon^{-1}\rceil}(\omega)>\tau$ for any $\varepsilon>0$, such that $|{}^o\alpha_t-{}^o\alpha_\tau|\le1/\lceil\varepsilon^{-1}\rceil\le\varepsilon$ on $[\tau(\omega),\tau_{1/\lceil\varepsilon^{-1}\rceil}(\omega))\ne\emptyset$. □

Appendix C. Supplementary results

Lemma C.1. *Suppose $\vartheta=\hat\tau_1^\vartheta=\hat\tau_2^\vartheta$ and $0=\alpha_i^\vartheta(\vartheta)<\alpha_j^\vartheta(\vartheta)<1$. Then $\frac{\lambda_M^\vartheta}{\lambda_{L,i}^\vartheta}=\frac{\alpha_j^\vartheta(\vartheta)}{1-\alpha_j^\vartheta(\vartheta)}$.*

Proof. We introduce the function $\mu_F(x,y):=\mu_L(y,x)=1-\mu_L(x,y)-\mu_M(x,y)$ and use the short-hand notation

$$\underline{\mu_\cdot}=\liminf_{\substack{t\searrow\vartheta\\ \alpha_i^\vartheta(t)+\alpha_j^\vartheta(t)>0}}\mu_\cdot(\alpha_i^\vartheta(t),\alpha_j^\vartheta(t)),\quad\overline{\mu_\cdot}=\limsup_{\substack{t\searrow\vartheta\\ \alpha_i^\vartheta(t)+\alpha_j^\vartheta(t)>0}}\mu_\cdot(\alpha_i^\vartheta(t),\alpha_j^\vartheta(t)),$$

$$\underline{\alpha_i}=\liminf_{t\searrow\vartheta}\alpha_i^\vartheta(t),\quad\overline{\alpha_i}=\limsup_{t\searrow\vartheta}\alpha_i^\vartheta(t)\quad\text{and}\quad\alpha_j=\alpha_j^\vartheta(\vartheta).$$

[40] ${}^o\alpha$ is the unique optional process such that $E[{}^o\alpha_\tau\mathbf{1}_{\{\tau<\infty\}}\,|\,\mathscr{F}_\tau]=\alpha_\tau\mathbf{1}_{\{\tau<\infty\}}$ a.s. for every $\tau\in\mathscr{T}$ (cf., e.g., Revuz and Yor, 1999, Theorem 5.6).

In our current case $\vartheta = \hat{\tau}_1^\vartheta = \hat{\tau}_2^\vartheta$ and $0 = \alpha_i^\vartheta(\vartheta) < \alpha_j^\vartheta(\vartheta) < 1$ we have

$$\lambda_{L,i}^\vartheta = (1-\alpha_j)\frac{1}{2}(\underline{\mu_L} + \overline{\mu_L}),$$
$$\lambda_{L,j}^\vartheta = \alpha_j + (1-\alpha_j)\frac{1}{2}(\underline{\mu_F} + \overline{\mu_F})$$

and thus

$$\begin{aligned}\lambda_M^\vartheta &= 1 - \lambda_{L,i}^\vartheta - \lambda_{L,j}^\vartheta \\ &= (1-\alpha_j)\frac{1}{2}(2 - \underline{\mu_L} - \overline{\mu_L} - \underline{\mu_F} - \overline{\mu_F}).\end{aligned}$$

Using $\alpha_j = \lim_{t\searrow\vartheta} \alpha_j^\vartheta(t)$ and the continuity and monotonicity of μ_L and μ_F we obtain

$$\begin{aligned}\frac{\lambda_M^\vartheta}{\lambda_{L,i}^\vartheta} &= \frac{2 - \underline{\mu_L} - \overline{\mu_L} - \underline{\mu_F} - \overline{\mu_F}}{\underline{\mu_L} + \overline{\mu_L}} \\ &= \frac{2 - \mu_L(\underline{\alpha_i}, \alpha_j) - \mu_L(\overline{\alpha_i}, \alpha_j) - \mu_F(\overline{\alpha_i}, \alpha_j) - \mu_F(\underline{\alpha_i}, \alpha_j)}{\mu_L(\underline{\alpha_i}, \alpha_j) + \mu_L(\overline{\alpha_i}, \alpha_j)} \\ &= \frac{\mu_M(\underline{\alpha_i}, \alpha_j) + \mu_M(\overline{\alpha_i}, \alpha_j)}{\mu_L(\underline{\alpha_i}, \alpha_j) + \mu_L(\overline{\alpha_i}, \alpha_j)} = \frac{\alpha_j}{1-\alpha_j}.\end{aligned}$$ □

Lemma C.2. *Fix $\sigma \in \mathscr{T}$ and suppose (G_1, α_1) and (G_2, α_2) are time-consistent extended mixed strategies which induce an equilibrium in all subgames Γ^ϑ beginning at some $\vartheta \in \mathscr{T}$ taking values in $(\sigma, \infty]$ a.s.*

If $\sigma = \hat{\tau}_1^\sigma = \hat{\tau}_2^\sigma$ and $\lim_{t\searrow\sigma} \alpha_1^\sigma(t) = \lim_{t\searrow\sigma} \alpha_2^\sigma(t) = 0$, then

$$L_\sigma^1 - F_\sigma^1 = L_\sigma^2 - F_\sigma^2 = 0 \qquad a.s.$$

Note that we do not impose right-continuity of any α_i^σ at σ in the lemma.

Proof. Let $i, j \in \{1, 2\}$, $i \neq j$. Suppose first $L_\sigma^i > F_\sigma^i$. By hypothesis there exist arbitrarily small right (random) neighbourhoods of σ in which α_i^σ is bounded away from 1, in which α_j^σ takes some strictly positive values, and in which $L^i > F^i\,(\geq M^i)$. In any such neighbourhood we must have by condition (3.1) that

$$\alpha_j^\sigma > 0 \Rightarrow \alpha_j^\sigma \geq \frac{L^i - F^i}{L^i - M^i},$$

implying $\limsup_{t\searrow\sigma} \frac{L^i - F^i}{L^i - M^i} = 0$ and therefore $L_\sigma^i = F_\sigma^i$.

Now suppose $L_\sigma^i < F_\sigma^i$. Then in any right (random) neighbourhood of σ in which $L^i < F^i$ and α_j^σ is bounded away from 1, α_i^σ can only be strictly positive where $\alpha_j^\sigma = 0$, i.e., the supports of α_i^σ and α_j^σ in these neighbourhoods must be disjoint. Hence, whenever $\alpha_i^\sigma > 0$, player i becomes leader and must prefer so over becoming follower at the next time when $\alpha_j^\sigma > 0$. Now consider neighbourhoods between σ and

$$\sigma' := \inf\{t \geq \sigma \,|\, L_t^i - L_\sigma^i \geq (F_\sigma^i - L_\sigma^i)/3 \text{ or } F_t^i - F_\sigma^i \leq -(F_\sigma^i - L_\sigma^i)/3\} > \sigma.$$

At any stopping time $\vartheta \in [\sigma, \sigma']$, i can only prefer to stop if $\alpha_j^\sigma = 0$ on $[\sigma, \sigma']$, which contradicts the hypothesis. $\square$

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